A cyclotron resonance mechanism for very-low-frequency whistler-mode sideband wave radiation. II. Description of internal ("trapped") resonances

L. A. D. Sá
Space, Telecommunications and Radioscience Laboratory, Stanford University, Stanford, California 94305

(Received 8 December 1988; accepted for publication 23 June 1989)

The second part of the paper presents an analytical and numerical treatment of cyclotron resonances formed inside the potential wells of a set of monochromatic carriers propagating in a duct in the magnetosphere. It is found that those resonances produce oscillatory disturbances in the potential well charge distribution that frequency modulate the carriers to produce sidebands. The frequency spectrum is the same as the one generated by the external resonances defined in part I. The carrier trapping frequencies do not affect the sideband wave frequencies and do not impose sharp constraints on the spectrum bandwidth, on the sideband line separation, or on the maximum interacting carrier frequency separation. When not highly distorted by wave growth, a two-carrier sideband spectrum should have an approximately exponential profile.

I. INTRODUCTION

In the second part of this three-part paper we will concentrate on the study and classification, together with an analysis of the sideband radiation they produce, of cyclotron resonances created by the electromagnetic fields of an incoming set of carriers acting on plasma electrons inside the carriers' own potential wells ("trapped" resonances). A previous reading of part I of this paper is recommended, since the same analytical methods are used to solve the equations describing the electron motion and the same type of phase plots are used for displaying numerical results. Most of he notation used in this second part is identical to the one used and defined in part I.

We begin part II by displaying, in Sec. II, "trapped" resonances with the help of phase plots, and discussing qualitatively the process of radiation formation. In Sec. III, as a preparation for the work to follow, the equations describing the electron motion are rewritten using action-angle variables which are more adequate for the study of motion inside the potential wells. The Kolmogorov–Arnold–Moser (KAM) theorem is applied in Sec. IV to the Hamiltonian expressed in action-angle variables and general expressions for the resonant frequencies of motion and associated radiation frequencies are obtained. In Sec. V, the Lie perturbation method is applied to solve the equation of motion for nonresonant electrons. Section VI describes analytically an infinite set of first-order resonances coming from the interaction of any wave pair in the incoming wave packet and radiating mainly at their first harmonic frequencies. Weaker radiation at the subharmonic frequencies is also obtained. Section VII describes two- and three-wave second-order effects. Two-wave effects consist of the second harmonic and a series of subharmonics. Three-wave effects contain intermodulation effects, together with the infinite series of their associated subharmonics. Section VIII describes sideband interference effects due to the direct interaction of resonances located inside a same wave potential. Section IX looks at higher-order effects, and Sec. X contains the conclusions.

II. PHASE PLOTS AND THE PROCESS OF RADIATION FORMATION

Figure 1 shows the time evolution of one of the trapped resonances we are going to study seen by an observer at a fixed point in space. The resonance is created by a main wave, inside whose potential well it oscillates, and by a weaker wave not visible in the plot. The resonance motion is phase locked to the weaker perturbing carrier and its rotation is at twice the carrier-carrier frequency difference as is indicated by the vector diagrams at the corner of the pictures (it is a second harmonic resonance). The trapping frequency of the main carrier is slightly higher than twice the two-carrier frequency difference, as should be expected, since the average rotation frequency of the resonant electrons is undisturbed by the presence of the resonance. A band of chaotic motion is seen around the resonance, and will be present in all cases where internal resonances are formed. Electrons in this chaotic band cannot be bunched, and its presence will contribute to the main carrier saturation by limiting its growth. We will see in the following sections that radiation from this trapped resonance will fall mainly outside the carrier potential well. Those two facts imply that, for this value of the trapping frequency, few of the electrons that are directly affected by the main wave will radiate at its frequency. In other words, for this electron population, the main wave has practically saturated due to the presence of the weaker perturbing carrier. On the other hand, the weaker carrier is located inside the main carrier potential well, one quarter of the way from the center to the separatrix, but no resonance directly associated with its presence is seen. This means that the weaker carrier is not able to bunch electrons on its own and grow. We say that, under those circumstances, the weak carrier growth has been suppressed by the strong carrier.

The radiation process can be understood if we remind ourselves, from part I, that resonances, coupled to the existence of an electron concentration gradient in $\mathbf{v}_{\parallel}$ at all points in space, can cause bunching when they act on the convective motion of the electrons moving along the duct. Observed at a

 0021-8979/89/203495-11$02.40

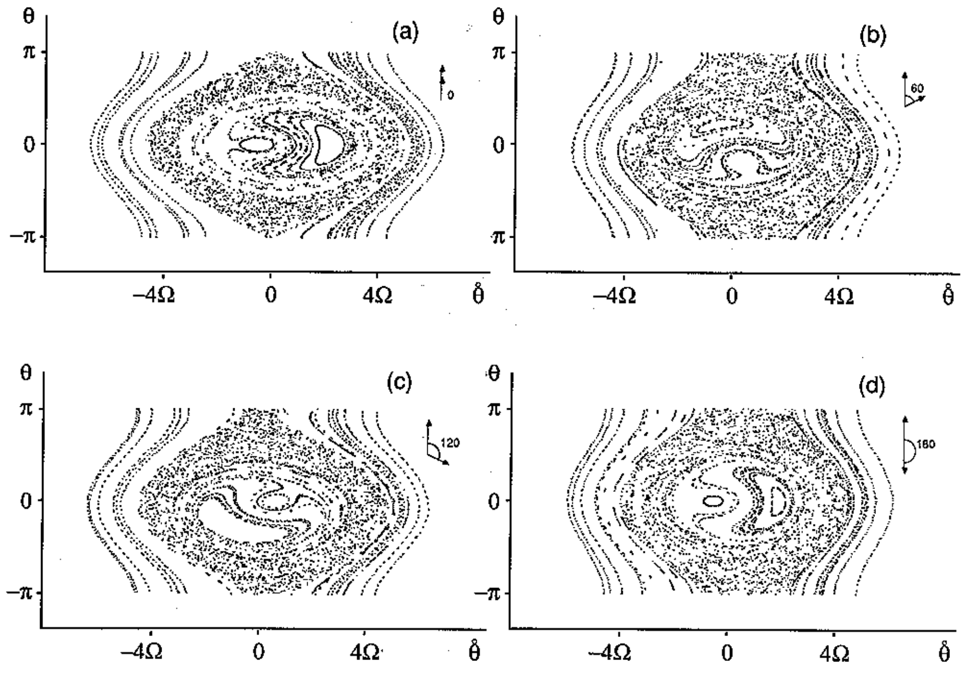

FIG. 1. Trapping resonance created by two interacting carriers. The main carrier is located at $\dot{\theta} = 0$ and the weaker, perturbing one at $\dot{\theta} = \Omega$. The main carrier trapping frequency is $\approx 2\Omega$, and the weak to main carrier amplitude ratio is 0.2. As the carriers rotate relatively to each other, the internal resonance rotates at twice their frequency separation: (a) Relative carrier phase = 0°. (b) Relative carrier phase = 60°. (c) Relative carrier phase = 120°. (d) Relative carrier phase = 180°.

fixed point in space, the trapped resonance will exhibit some bunching, causing a localized perturbation inside the potential well. Now, the resonance is phase locked to the weak carrier, and as time flows, its position, together with the perturbation in electron concentration associated with it, will oscillate inside the main carrier potential well. This oscillation will modulate in frequency the radiation produced by the main carrier, creating sidebands. An important fact to be noticed is that the radiation process is a collective phenomenon coming from oscillations of distorted electron concentrations inside the potential well, occurring as a function of time at a fixed point in space, independently of the existence of full oscillations for individual electrons as they move along the duct.

Figures 2(a) and 2(b) have a pictorial description of the resonance radiation mechanism. We assume that each internal resonance, due to the disturbances it creates in the electron concentration, constitutes a localized source of radiation, instantaneously monochromatic. If the carrier has frequency ω_1 and the resonance turns inside the carrier wave potential with frequency $\Delta\omega$, radiation will be observed at frequencies $\omega = \omega_1 \pm n\Delta\omega$, $0 \leqslant n < \infty$ as a consequence of the frequency modulation process.

Figures 2(c) and 2(d) show the case when the internal resonance has two lobes. Each lobe will radiate the same frequency spectrum shifted in phase by 180°. The modulation around the carrier frequency will now repeat itself twice as fast, creating a spectrum of lines separated from each oth-

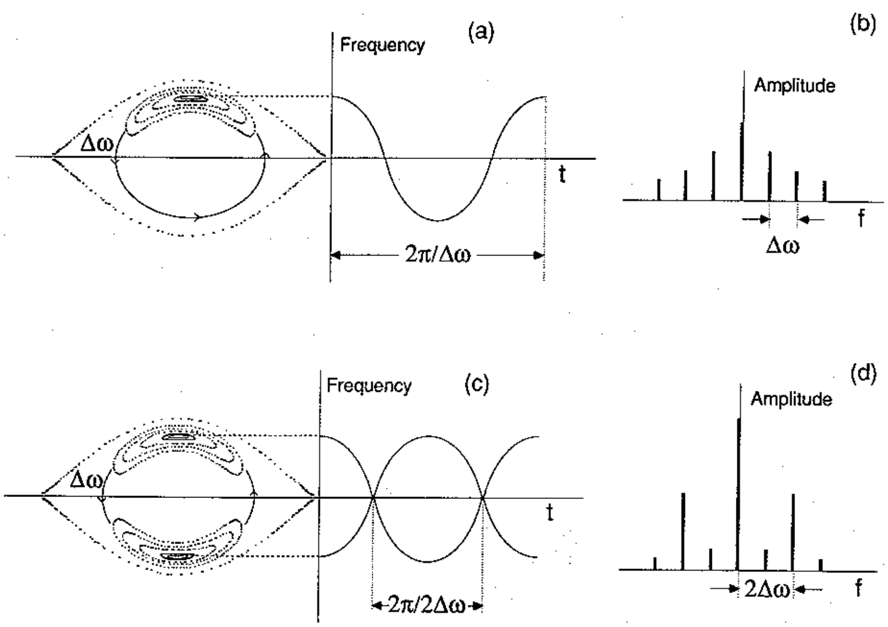

FIG. 2. Radiation from oscillating trapped resonance: (a) As the resonance oscillates with frequency $\Delta\omega$ in the potential well, it creates variable frequency radiation with period $2\pi/\Delta\omega$. (b) Demodulation shows a spectrum of lines separated by $\Delta\omega$. (c) If the resonance is two-lobed, the radiation will have a period of $2\pi/(2\Delta\omega)$. (d) Demodulation will show lines separated by a frequency $2\Delta\omega$. Since the two resonance populations are not identical, weak half harmonics will be found between any two main harmonics.

er by the frequency $2\Delta\omega$ if the perturbations in electron concentration are exactly the same for the two lobes. If some difference occurs, due, e.g., to variable distortions caused by the perturbing carrier, the spectrum will also have odd multiples of $\Delta\omega$ which will show up as weak half harmonics between the main even multiples of $\Delta\omega$.

III. ELECTRON EQUATION OF MOTION IN ACTION-ANGLE VARIABLES

From part I, we know that the equations describing the electron motion under the influence of N waves are

$$\ddot{\theta} = -\sum_{i=1}^{N} \Omega_{ti}^2 \sin(\theta - \Omega_i t + \phi_i), \quad (1)$$

$$\Omega_i = [v_\parallel / v_g + 1]\Delta\omega_i, \quad (2)$$

$$\phi_i = (\Omega_i - \Delta\omega_i)t_0 + \Delta\phi_i, \quad (3)$$

where $\Omega_{ti}^2 = ev_\perp k_1 B_i/m$ is the square of the trapping frequency associated with wave i, Ω_i is the Doppler-shifted frequency difference between wave i and reference wave 1, and v_g is the wave group velocity at the average radiation frequency.

For the study of internal resonances created by the wave-wave interaction process, it will be extremely convenient to single out one of the waves that can always be wave 1 by an appropriate labeling of the carriers, and rewrite the equation of motion using (j,ϕ), action-angle variables for electron motion under the influence of that single wave. The meaning of (j,ϕ) can be seen from Fig. 3 and the defining equations:

$$j = \frac{1}{2\pi}\oint \dot{\theta}\, d\theta = \frac{2\theta_{\max}}{\pi}\langle\dot{\theta}\rangle. \quad (4)$$

j is proportional to the average value of $\dot{\theta}$ over the electron phase space trajectory and equal to the area enclosed by the trajectory divided by 2π. ϕ is defined from the main wave trapping oscillation frequency:

$$\phi = \Omega_t(j)(t - t_0). \quad (5)$$

ϕ goes through one cycle in the same time the electron goes through one oscillation inside the potential well, and can be thought of as describing the phase of the motion. By an appropriate choice of t_0, the relation between θ and ϕ is

$$\theta = 2\sin^{-1}\left[\kappa\, \mathrm{sn}\left(\frac{2K(\kappa)\phi}{\pi}\right)\right], \quad (6)$$

with sn the Jacobian sine function and K the complete elliptic integral of the first kind. κ is defined by

$$\kappa^2 = (1 + h_0/\Omega_{t1}^2)/2, \quad (7)$$

h_0 being the energy of the motion. j is given by

$$j = (8\Omega_{t1}/\pi)[E(\kappa) - (1-\kappa^2)K(\kappa)], \quad (8)$$

where E is the complete elliptic integral of the second kind. The trapping frequency can be expressed as

$$\Omega_t(j) = \Omega_{t1}\pi/2K(\kappa), \quad 0 \leqslant \Omega_t(j) \leqslant \Omega_{t1}. \quad (9)$$

Written in Hamiltonian form the equation of motion becomes

$$h = \frac{\dot{\theta}^2}{2} - \Omega_{t1}^2 \cos\theta - \sum_{i=2}^{N} \Omega_{ti}^2 \cos(\theta - \Omega_i t + \phi_i). \quad (10)$$

As a function of (j,ϕ), it can be rewritten as

$$h(j,\phi) = h_0(j) - \sum_{i=2}^{N} \Omega_{ti}^2 \cos[\theta(\phi) - \Omega_i t + \phi_i]. \quad (11)$$

The cos terms inside the summation can be explicitly written as a function of (j,ϕ). This was done by Smith and Pereira[1] and the result is reproduced here:

$$\cos[\theta(\phi) - \Omega_i t + \phi_i] = \sum_{n=-\infty}^{\infty} V_n(j)\cos(n\phi - \Omega_i t + \phi_i), \quad (12)$$

with

$$V_n(j) = \left(\frac{\pi}{K(\kappa)}\right)^2 \frac{nq^{n/2}}{1-(-q)^n} \quad (n \neq 0), \quad (13)$$

$$V_0(j) = 2E(\kappa)/K(\kappa) - 1, \quad (14)$$

and $q = \exp[-\pi K(\sqrt{1-\kappa^2})/K(\kappa)]$. An important property of the $V_n(j)$, is that $V_n(j) = (-1)^n V_{-n}(j)$. Figure 4 contains plots of V_n as a function of κ^2 for a few values of n.

The Hamiltonian can be finally written as

$$h(j,\phi) = h_0(j) - \sum_{i=2}^{N}\sum_{n=-\infty}^{\infty} \Omega_{ti}^2 V_n(j)\cos(n\phi - \Omega_i t + \phi_i). \quad (15)$$

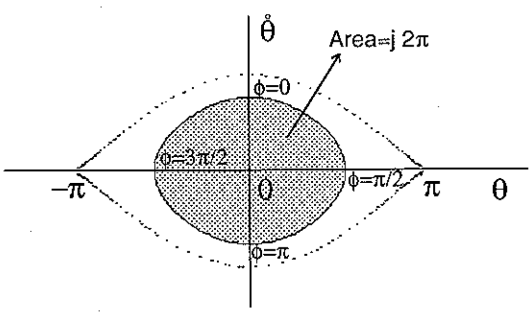

FIG. 3. Action-angle variables used for description of trapped resonances. $2\pi j$ is the area enclosed by the phase trajectory and ϕ is the phase of the motion. Although $\dot{\phi}$ is a constant, $\dot{\theta}$ has a complex dependence on time.

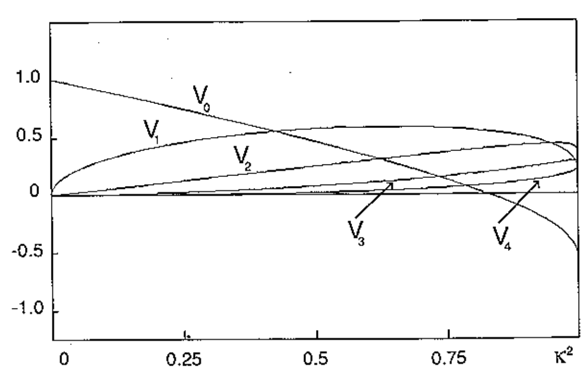

FIG. 4. Plots of the potential V_n as a function of κ^2 for $n = 0...4$. Negative values of n can be obtained by noticing that $V_{-n}(j) = (-1)V_n(j)$.

IV. GENERAL SOLUTION OF THE EQUATION OF MOTION–KAM THEOREM

As in part I, the system we are studying has $N+1$ degrees of freedom: the phase of the N waves, and the electron angular position. But since we have arbitrarily defined the origin of frequencies as being wave 1, we are left with only an N-degree of freedom system that can be described by the N-dimensional Hamiltonian:

$$h_N(q_i,p_i) = h_0(p_1) + \sum_{i=2}^{N} \Omega_i p_i - \sum_{i=2}^{N} \sum_{n=-\infty}^{\infty} \Omega_{ti}^2 V_n(p_1)\cos(nq_1 - q_i + \phi_i), \tag{16}$$

with the following equations of motion:

$$\dot{q}_1 = \frac{\partial h_N}{\partial p_1} = \frac{\partial h_0}{\partial p_1} - \sum_{i=2}^{N} \sum_{n=-\infty}^{\infty} \Omega_{ti}^2 V_n'(p_1) \times\cos(nq_1 - q_i + \phi_i), \tag{17}$$

$$\dot{p}_1 = -\frac{\partial h_N}{\partial q_1} = -\sum_{i=2}^{N} \sum_{n=-\infty}^{\infty} n\Omega_{ti}^2 V_n(p_1)\sin(nq_1 - q_i + \phi_i), \tag{18}$$

$$\dot{q}_i = \frac{\partial h_N}{\partial p_i} = \Omega_i \quad (i\neq 1), \tag{19}$$

and

$$\dot{p}_i = -\frac{\partial h_N}{\partial q_i} = \sum_{n=-\infty}^{\infty} \Omega_{ti}^2 V_n(p_1)\sin(nq_1 - q_i + \phi_i) \quad (i\neq 1). \tag{20}$$

The p_i for $i\neq 1$ are dummy variables and can be ignored. The equations for the q_i give

$$q_i = \Omega_i t. \tag{21}$$

Substituting those values in Eqs. (17) and (18), making $p_1 = j$, and $q_1 = \phi$, we get the right equation of motion. If we now define

$$H_0(p_i) = h_0(p_1) + \sum_{i=2}^{N} \Omega_i p_i \tag{22}$$

and

$$H_1(q_i,p_i) = -\sum_{i=2}^{N} \sum_{n=-\infty}^{\infty} \Omega_{ti}^2 V_n(p_1) \times\cos(nq_1 - q_i + \phi_i), \tag{23}$$

we can write

$$h_N(q_i,p_i) = H_0(p_i) + H_1(q_i,p_i), \quad i=1\ldots N, \tag{24}$$

where $H_1(q_i,p_i)$ is periodic in all q_i, and $H_0(p_i)$ depends only on the p_i. Once again, the KAM theorem for nonlinear systems states that the motion described by H_0 will be appreciably affected by H_1 only if the following relationship holds for the variables describing the unperturbed motion:

$$\sum_{i=1}^{N} n_i \dot{q}_i = 0, \tag{25}$$

n_i being arbitrary integers. Making $\dot{q}_i = \dot{\phi} = \Omega$, we will have

$$\Omega = \sum_{i=2}^{N} n_i \Omega_i / n. \tag{26}$$

That is, the bounce (trapping) frequency of the electron resonates with the Doppler-shifted wave frequency differences. Since we have already fixed the origin at wave 1, there are no free parameters left and no additional constraint is imposed on n.

To see the effect of the resonances at a fixed point in space, and derive an expression for the radiation frequencies, we need to define an electron stream. For that, we take a set of electrons with trapping frequency Ω and an infinitesimally weak wave with a Doppler-shifted frequency offset also equal to Ω. Such a wave is said to be resonant with the electrons because their relative phases are constant. Next we define the stream as a subset of those electrons having a chosen fixed phase relative to the electromagnetic wave at all points in space, and consequently at all instants of time. Since the wave at a fixed point in space turns with frequency $\Delta\omega = \Omega v_g/(v_\parallel + v_g)$, so will the phase ϕ of the electrons in the stream, which can now be written as

$$\phi = \Omega(t - t_0) + \Delta\omega t_0 + \phi_0, \tag{27}$$

where t_0 is the moment the electron at $z = -v_\parallel (t - t_0)$ enters the interaction region. The radiation coming from such a stream can be substituted for the radiation coming from the accelerated electrons. Since $\Delta\omega_i = \Omega_i v_g/(v_\parallel + v_g)$, the expression for the resonance frequencies can be rewritten as

$$\Delta\omega = \sum_{i=2}^{N} n_i \Delta\omega_i / n \tag{28}$$

where $\Delta\omega$ is the oscillation frequency of a cross section of the stream we have just defined. The electron stream oscillations will frequency modulate the reference wave monochromatic radiation, as mentioned in Sec. II, adding lines at $\omega = \omega_1 + m\Delta\omega$, m being an arbitrary integer. The radiation frequencies will then be

$$\omega = \omega_1 + \frac{m}{n} \sum_{i=2}^{N} n_i \Delta\omega_i. \tag{29}$$

Since $\Delta\omega_i = \omega_i - \omega_1$ and n_i can be substituted for mn_i because both are arbitrary integers, we can write

$$\omega = \left[\omega_1\left(n - \sum_{i=2}^{N} n_i\right) + \sum_{i=2}^{N} n_i\omega_i\right] \Big/ n. \tag{30}$$

Defining $n_1 = (n - \Sigma_{i=2}^{N} n_i)$, we finally get

$$\omega = \sum_{i=1}^{N} n_i\omega_i \Big/ \sum_{i=1}^{N} n_i, \tag{31}$$

which is the same formula for the frequencies as obtained in part I.

V. ELECTRON MOTION FOR NONRESONANT VALUES OF $V_\parallel$

If we use the explicitly time-dependent form of the Hamiltonian, and put $\Omega_{ti}^2 = \epsilon A_i$, we have

$$h(\phi,j,t) = h_0(\phi,j,t) + \epsilon h_1(\phi,j,t), \tag{32}$$

with

$$h_0 = h_0(j),$$

$$h_1 = -\sum_{i=2}^{N} \sum_{n=-\infty}^{\infty} A_i V_n(j)\cos(n\phi - \Omega_i t + \phi_i). \tag{33}$$

To study nonresonant terms, using the Lie perturbation method described in part I, we ignore all resonances, and put in the K_n only constant terms. To get the simplest nontrivial result we have to go up to second order in the perturbation expansion. Doing this, we get $K_0 = h_0$ and $K_1 = 0$ (because h_1 has no constant terms).

This implies

$$W_1 = -\int^t d\tau\, h_1[\Phi + \Omega_t(\tau - t),J,\tau]$$

$$= \sum_{i=2}^{N} \sum_{n=-\infty}^{\infty} \frac{A_i V_n(J)}{n\Omega_t(J) - \Omega_i} \sin(n\Phi - \Omega_i t + \phi_i). \tag{34}$$

We then choose

$$2K_2 = \langle L_1 h_1\rangle = \langle\{W_1,h_1\}\rangle. \tag{35}$$

A straightforward calculation shows that

$$\{W_1,h_1\} = \frac{\partial W_1}{\partial \Phi}\frac{\partial h_1}{\partial J} - \frac{\partial W_1}{\partial J}\frac{\partial h_1}{\partial \Phi}$$

$$= \frac{1}{2}\sum_{i,j=2}^{N}\sum_{m,n=-\infty}^{\infty} A_j(ma'_{in}V_m - na_{in}V'_m)$$
$$\times\cos[(n+m)\Phi - (\Omega_i + \Omega_j)t + \phi_i + \phi_j]$$
$$-\frac{1}{2}\sum_{i,j=2}^{N}\sum_{m,n=-\infty}^{\infty} A_j(ma'_{in}V_m + na_{in}V'_m)$$
$$\times\cos[(n-m)\Phi - (\Omega_i - \Omega_j)t + \phi_i - \phi_j], \tag{36}$$

where

$$a_{in} = \frac{A_i V_n(J)}{n\Omega_t(J) - \Omega_i}, \tag{37}$$

and the prime indicates differentation with respect to J. If $\Omega_i \neq -\Omega_j$ (no symmetrically placed waves in the system),

$$K_2 = K_2^0 = \tfrac{1}{2}\langle\{W_1,h_1\}\rangle$$

$$= -\frac{1}{4}\sum_{i,n} A_i(na'_{in}V_n + na_{in}V'_n)$$

$$= -\frac{1}{4}\sum_{i,n} A_i n(a_{in}V_n)' \tag{38}$$

and

$$K = h_0(J) - \frac{\epsilon^2}{4}\sum_{i,n}\frac{nA_i^2V_n^2}{n\Omega_t - \Omega_i}\left(\frac{2V'_n}{V_n} - \frac{n\Omega'_t}{n\Omega_t - \Omega_i}\right). \tag{39}$$

Hamilton's equations of motion for K can be solved:

$$j = -\frac{\partial K}{\partial \Phi} = 0, \tag{40}$$

$$\dot\Phi = \frac{\partial K}{\partial J} = \frac{\partial h_0(J)}{\partial J} + \epsilon^2\frac{\partial K_2^0}{\partial J} = \Omega. \tag{41}$$

The solution is

$$J = J_0, \tag{42}$$

$$\Phi = \Omega t + \Phi_0 = \Omega_t(J_0)t + \epsilon^2\frac{\partial K_2^0}{\partial J}t + \Phi_0. \tag{43}$$

The transformation back to (ϕ,j), keeping only the first-order terms and constant second-order terms, is given by

$$\phi = \Phi - \epsilon\frac{\partial W_1}{\partial J} - \frac{\epsilon^2}{2}\left\langle\left\{W_1,\frac{\partial W_1}{\partial J}\right\}\right\rangle. \tag{44}$$

$$j = J + \epsilon\frac{\partial W_1}{\partial \Phi} + \frac{\epsilon^2}{2}\left\langle\left\{W_1,\frac{\partial W_1}{\partial \Phi}\right\}\right\rangle. \tag{45}$$

Evaluation of the Poisson brackets gives

$$\left\langle\left\{W_1,\frac{\partial W_1}{\partial J}\right\}\right\rangle = 0 \tag{46}$$

and

$$\left\langle\left\{W_1,\frac{\partial W_1}{\partial \Phi}\right\}\right\rangle = \sum_{i,n} A_i n^2 a_{in} a'_{in}, \tag{47}$$

which gives

$$\phi = \Phi - \epsilon\sum_{i,n}\frac{A_iV_n}{n\Omega_t - \Omega_i}\left(\frac{V'_n}{V_n} - \frac{n\Omega'_t}{n\Omega_t - \Omega_i}\right)$$
$$\times\sin(n\Phi - \Omega_i t + \phi_i), \tag{48}$$

$$j = J + \frac{\epsilon^2}{2}\sum_{i,n}\frac{n^2A_i^2V_n^2}{(n\Omega_t - \Omega_i)^2}\left(\frac{V'_n}{V_n} - \frac{n\Omega'_t}{n\Omega_t - \Omega_i}\right)$$
$$+\epsilon\sum_{i,n}\frac{nA_iV_n}{n\Omega_t - \Omega_i}\cos(n\Phi - \Omega_i t + \phi_i). \tag{49}$$

We see that up to zeroth order (j,ϕ) is equal to (J,Φ), and the motion is just the normal oscillation, with frequency $\Omega_t(J_0)$, of the electron inside the potential well.

The first-order oscillatory terms in the above equations predict the existence of a conditionally periodic jittering motion superimposed on the main motion, maximized when $n\Omega_t \approx \Omega_i$ for some i and n. Under those conditions, the particle motion will add to the bunching associated with such a resonance.

The second-order corrections describe an important effect, here due to the adiabatic invariance of J: If we average away the oscillatory terms, and assume that a certain particle has $j = J_0$ for $\epsilon = 0$, and that the wave fields are adiabatically turned on, then J will remain constant, and ϕ will vary as

$$\dot\phi = \Omega_t(J_0) + \epsilon^2\frac{\partial K_2^0(J_0)}{\partial J_0}. \tag{50}$$

If J_0 is such that $n\Omega_t(J_0) \approx \Omega_i$, for a certain i,

$$j = J_0 + \Delta j \approx J_0 - \frac{\epsilon^2}{2}\frac{n^3A_i^2V_n^2\Omega'_t}{(n\Omega_t - \Omega_i)^3}. \tag{51}$$

and

$$\dot{\phi} \approx \Omega_t(J_0) + \Omega_t' \Delta j \approx \Omega_t(J_0) - \frac{\epsilon^2}{2} \frac{n^3 A_i^2 V_n^2 \Omega_t'^2}{(n\Omega_t - \Omega_i)^3}. \quad (52)$$

The particle will be shifted due to the presence of the waves, the forces being such that each wave tends to pull the particle towards points where the wave Doppler-shifted frequency offset is a multiple of the particle trapping frequency. This shift will create a gradient around those points, increasing the growth rate of sideband waves associated with resonances located at those frequencies. (Those will be the first-order resonances described in the next section.)

Since radiation is a collective phenomenon, it will be extremely convenient to observe the effects of the electron motion, using a stream defined according to the outlines established in Sec. IV. The whole electron distribution can be broken down into such streams, their bunching and distortion determining the radiation characteristics of the system. Those streams will also be the loci of the equilibrium points for the resonances to be studied in the following sections. The stream will be composed of electrons with trapping frequency Ω, and its cross section will oscillate at the frequency $\Delta\omega = \Omega v_g/(v_\parallel + v_g)$. The phase of the component electrons can be obtained by conveniently defining Φ_0 in the general solution for Φ:

$$\Phi = \Omega t + \Phi_0 = \Omega(t - t_0) + \Delta\omega t_0 + \Delta\phi. \quad (53)$$

The electrons in such a stream have equal radiation characteristics, because they all describe the same path in the wave potential (with the right phase delays to account for the finite radiation propagation velocity). The stream forms a rigid direct current (dc) line consisting of a spatially periodic distorted straight line wound around a spiral path, and its motion can be studied in a variety of ways. The most convenient is to observe the motion of its cross section at a fixed point in space. From the above equation it can be seen that such a motion will be oscillatory with frequency $\Delta\omega$, producing radiation at multiples of the oscillation frequency. We see that individual electron motion after integration over the stream becomes only indirectly relevant. The stream can be so short that no electron in it will have time to perform a full oscillation in the potential well. However, the stream can still continuously execute full oscillations, radiating at well-defined frequencies, if the electron flow is never interrupted.

VI. FIRST HARMONIC RESONANCE AND ASSOCIATED SUBHARMONICS

To study first-order resonances we put

$$K_0 = h_0 \quad (54)$$

and choose

$$K_1 = -A_j V_m(J)\cos(m\Phi - \Omega_j t + \phi_j) \quad (55)$$

if $\Omega_i \neq -\Omega_j$, for all i. This implies

$$W_1 = \sum_{\substack{(i\neq j)\text{ or}\\(n\neq m)}} \frac{A_i V_n(J)}{n\Omega_t(J) - \Omega_i} \sin(n\Phi - \Omega_i t + \phi_i) \quad (56)$$

and

$$K = h_0(J) - \epsilon A_j V_m(J)\cos(m\Phi - \Omega_j t + \phi_j). \quad (57)$$

Hamilton's equations applied to K will give

$$\dot{J} = -\frac{\partial K}{\partial \Phi} = -\epsilon m A_j V_m(J)\sin(m\Phi - \Omega_j t + \phi_j), \quad (58)$$

$$\dot{\Phi} = \frac{\partial K}{\partial J} = \frac{\partial h_0(J)}{\partial J} - \epsilon A_j V_m'(J)\cos(m\Phi - \Omega_j t + \phi_j), \quad (59)$$

$$\ddot{\Phi} \approx \frac{\partial^2 h_0(J)}{\partial J^2}\dot{J} = -\epsilon \frac{\partial^2 h_0(J)}{\partial J^2} m A_j V_m(J) \times \sin(m\Phi - \Omega_j t + \phi_j). \quad (60)$$

Defining

$$\Delta\Phi = m\Phi - \Omega_j t + \phi_j, \quad (61)$$

we can write the equation of motion as

$$\Delta\ddot{\Phi} = -\epsilon m^2 A_j \frac{\partial^2 h_0(J)}{\partial J^2} V_m(J)\sin(\Delta\Phi). \quad (62)$$

This resonance has a full width, in the $\dot{\Phi}$ variable, equal to

$$\Gamma_1 = 4\left(\epsilon A_j \left|\frac{\partial^2 h_0(J)}{\partial J^2} V_m(J)\right|\right)^{1/2}. \quad (63)$$

The width in J is obtained by noticing that

$$\dot{\Phi} = \Omega_{t1} - J/8 + \ldots, \quad (64)$$

which implies $\Gamma_1(J) = 8\Gamma_1(\dot{\Phi})$.

If $m<0$ and odd, the equilibrium points will be $\Delta\Phi = 2k\pi$, otherwise they will be given by $\Delta\Phi = (2k+1)\pi$. The electrons at the equilibrium points will oscillate with frequency $\dot{\Phi}$ subject to the following constraints:

$$0 \leqslant \dot{\Phi} = \Omega_j/m \leqslant \Omega_{t1}, \quad (65)$$

which gives a minimum value for $|m|$ for a given wave trapping frequency:

$$|m| \geqslant |\Omega_j|/\Omega_{t1}. \quad (66)$$

Using the expression for ϕ_j from Eq. (3), we get for the motion of the equilibrium streams:

$$\Phi = \frac{\Delta\omega_j}{m} t_0 + \frac{\Omega_j(t - t_0)}{m} + \frac{(2k+1)\pi - \Delta\phi}{m}, \quad k = 0\ldots(|m| - 1). \quad (67)$$

The equation describes a $|m|$-lobed resonance oscillating inside the potential well at a fixed point in space with frequency $\Delta\omega_j/m$. To look at the stream motion in the (j,ϕ) variables, we need the transformation equations:

$$\phi = \Phi - \epsilon \sum_{\substack{(i\neq j)\text{ or}\\(n\neq m)}} \frac{A_i V_n}{n\Omega_t - \Omega_i}\left(\frac{V_n'}{V_n} - \frac{n\Omega_t'}{n\Omega_t - \Omega_i}\right) \times \sin(n\Phi - \Omega_i t + \phi_i) \quad (68)$$

$$j = J + \frac{\epsilon^2}{2} \sum_{\substack{(i\neq j)\text{ or}\\(n\neq m)}} \frac{n^2 A_i^2 V_n^2}{(n\Omega_t - \Omega_i)^2}\left(\frac{V_n'}{V_n} - \frac{n\Omega_t'}{n\Omega_t - \Omega_i}\right) + \epsilon \sum_{\substack{(i\neq j)\text{ or}\\(n\neq m)}} \frac{n A_i V_n}{n\Omega_t - \Omega_i}\cos(n\Phi - \Omega_i t + \phi_i). \quad (69)$$

Since $m\Omega_t = \Omega_j$ and $m\Phi - \Omega_j t + \phi_j = (2k+1)\pi$, we can put

$$n\Phi - \Omega_i t + \phi_i = \frac{(n\Delta\omega_j - m\Delta\omega_i)}{m} t_0 + \Phi_{ijnmk}. \quad (70)$$

From the equations of motion we can also get

$$\Omega_t = \Omega_j/m = \Omega_{t1} - J/8 + \epsilon A_j V'_m. \quad (71)$$

Combining those equations, we have the expressions for the coordinate transformations:

$$\phi = \frac{\Delta\omega_j}{m} t_0 + \Phi_{0k} - \epsilon \sum_{\substack{(i\neq j) \text{ or} \\ (n\neq m)}} \frac{mA_i V_n}{n\Omega_j - m\Omega_i} \times \left(\frac{V'_n}{V_n} - \frac{mn\Omega'_t}{n\Omega_j - m\Omega_i}\right) \times \sin\left(\frac{(n\Delta\omega_j - m\Delta\omega_i)}{m} t_0 + \Phi_{ijnmk}\right), \quad (72)$$

$$j = 8(\Omega_{t1} - \Omega_j/m) + 8\epsilon A_j V'_m + \epsilon \sum_{\substack{(i\neq j) \text{ or} \\ (n\neq m)}} \frac{nmA_i V_n}{n\Omega_j - m\Omega_i} \times \cos\left(\frac{(n\Delta\omega_j - m\Delta\omega_i)}{m} t_0 + \Phi_{ijnmk}\right). \quad (73)$$

The equations show that points in the resonance rotate with an average frequency $\Delta\omega_j/m$ and are slightly shifted in j towards regions of higher potential, V_m. This rotation in phase space is equivalent to oscillations around the mean rotation frequency of the electron stream in real space, creating a frequency modulation of the radiation coming from the carrier. The resulting radiation spectrum has lines with frequencies $\omega_1 + (n/m)\Delta\omega_j$, $-\infty < n < \infty$, for a single lobe. The multiple lobed structure changes the spectrum appreciably. If all lobes have exactly the same electron distribution, the system will repeat itself at $m(\Delta\omega_j/m)$ and the spectrum will have frequencies $\omega_1 + n\Delta\omega_j$. Since different lobes will have somewhat different electron distributions, subharmonic radiation will show up at frequencies $\omega_1 + (nk/m)\Delta\omega_j$, $1 \leqslant k < m$. Since the differences in electron population will not be very large, those subharmonics will in general have smaller amplitudes than the integer harmonics of $\Delta\omega_j$.

Since the maximum frequency shift in the instantaneous radiation frequency, $\Delta\omega_{\text{shift}}$, is equal to $2\Delta\omega_{t1}$, the resonance half-width that the electron stream sees, and $\Delta\omega \leqslant \Delta\omega_{t1}$, where $\Delta\omega_{t1} = \Omega_{t1} v_g/(v_\parallel + v_g)$ is the carrier trapping frequency for the electron stream, the modulation index, $\Delta\omega_{\text{shift}}/\Delta\omega$, can easily be of the order of 1, generating quite a wideband spectrum.

If $n\Omega_j \neq m\Omega_i$ for all i and n, the time-dependent distortions of order ϵ present in the transformation equations for (j,ϕ) will give contributions of order $\epsilon \cdot \epsilon^{1/2}$ to the radiation and will be negligible up to the second-order effects we will be analyzing in this paper. They might be important, however, when compared to higher-order terms. If, on the other hand $n\Omega_j \approx m\Omega_i$ for some i and n, then resonance overlaps are present and the choice for K_1 has not been correctly made.

VII. SECOND-ORDER RESONANCES

A. General expressions

To look for second-order resonances, we choose K_0', K_1, and W_1 to be the same as in the nonresonant case. $2K_2$ will be obtained from terms in $\{W_1,h_1\}$:

$$\{W_1,h_1\} = \frac{1}{2} \sum_{i,j=2}^{N} \sum_{m,n=-\infty}^{\infty} A_j(ma'_{in} V_m - na_{in} V'_m) \times \cos[(n+m)\Phi - (\Omega_i + \Omega_j)t + \phi_i + \phi_j] - \frac{1}{2} \sum_{i,j=2}^{N} \sum_{m,n=-\infty}^{\infty} A_j(ma'_{in} V_m + na_{in} V'_m) \times \cos[(n-m)\Phi - (\Omega_i - \Omega_j)t + \phi_i - \phi_j]. \quad (74)$$

K_2 will always consist of K_0^2, the constant term obtainable from $\{W_1,h_1\}$, plus a chosen resonant term. In all cases, K_2 can be written as

$$K_2 = K_2^0 + K_2^1 \cos(l\Phi - \Omega_{ij} t + \phi_{ij}) \quad (75)$$

and

$$K = h_0 + \epsilon^2 K_2^0 + \epsilon^2 K_2^1 \cos(l\Phi - \Omega_{ij} t + \phi_{ij}), \quad (76)$$

with $\phi_{ij} = (\Omega_{ij} - \Delta\omega_{ij})t_0 + \Delta\phi_{ij}$.

Hamilton's equations will be

$$\dot{J} = -\frac{\partial K}{\partial \Phi} = \epsilon^2 l K_2^1 \sin(l\Phi - \Omega_{ij} t + \phi_{ij}), \quad (77)$$

$$\dot{\Phi} = \frac{\partial K}{\partial J} = \frac{\partial h_0(J)}{\partial J} + \epsilon^2 K_2^{0\prime} + \epsilon^2 K_2^{1\prime} \cos(l\Phi - \Omega_{ij} t + \phi_{ij}), \quad (78)$$

$$\ddot{\Phi} \approx \frac{\partial^2 h_0(J)}{\partial J^2} \dot{J} = \epsilon^2 l \frac{\partial^2 h_0(J)}{\partial J^2} K_2^1 \sin(l\Phi - \Omega_{ij} t + \phi_{ij}). \quad (79)$$

Defining

$$\Delta\Phi = l\Phi - \Omega_{ij} t + \phi_{ij}, \quad (80)$$

we can write the equation of motion as:

$$\Delta\ddot{\Phi} = \epsilon^2 l^2 \frac{\partial^2 h_0(J)}{\partial J^2} K_2^1 \sin(\Delta\Phi). \quad (81)$$

The resonances have full widths, in the $\dot{\Phi}$ variable, equal to

$$\Gamma_- = 4\epsilon \left(\left|\frac{\partial^2 h_0(J)}{\partial J^2} K_2^1\right|\right)^{1/2}. \quad (82)$$

Since $\partial^2 h_0(J)/\partial J^2 < 0$, the equilibrium values of $\Delta\Phi$ will be

$$\Delta\Phi_0 = (2k+1)\pi \quad \text{if} \quad K_2^1 < 0, \quad (83)$$

$$\Delta\Phi_0 = 2k\pi \quad \text{if} \quad K_2^1 > 0, \quad (84)$$

with $k = 0...(|l| - 1)$. For those equilibrium points, which define $|l|$-lobed resonances, Φ will vary as

$$\Phi = \frac{\Omega_{ij} t - \phi_{ij} + \Delta\Phi_0}{l}, \quad (85)$$

which can be rewritten as

$$\Phi = \frac{\Delta\omega_{ij} t_0}{l} + \frac{\Omega_{ij}(t - t_0)}{l} - \frac{\Delta\phi_{ij} - \Delta\Phi_0}{l}. \quad (86)$$

The resonances will oscillate in the potential well with frequencies

$$\Delta\omega = \Delta\omega_{ij}/l, \quad (87)$$

and will radiate at the frequencies

$$\omega = \omega_1 + (n\Delta\omega_{ij}/l), \quad (88)$$

with enhanced amplitudes at frequencies such that n/l is an integer.

B. Three-wave resonances: Intermodulation

1. Frequency subtraction

If we take two distinct waves with frequencies Ω_i and Ω_j, the term in $\{W_1, h_1\}$ asymmetrical under permutation of the indices i and j gives

$$4(K_2 - K_2^0) = -\sum_{n-m=l}(\beta_{ijmn} + \beta_{jinm}) \times\cos[l\Phi - (\Omega_i - \Omega_j)t + \phi_i - \phi_j], \quad (89)$$

with

$$\beta_{ijmn} = A_j(ma'_{in}V_m + na_{in}V'_m). \quad (90)$$

Calculation of the β's leads to the following expression for K_2:

$$K_2 - K_2^0 = \frac{A_iA_j}{2}\sum_{n=-\infty}^{\infty}\frac{lV_nV_{n-l}(n-l)n}{(l-n)\Omega_i + n\Omega_j} \times\left(\frac{V'_n}{nV_n} + \frac{V'_{n-l}}{(n-l)V_{n-l}} + \frac{l\Omega'_i}{(l-n)\Omega_i + n\Omega_j}\right) \times\cos[l\Phi - (\Omega_i - \Omega_j)t + \phi_i - \phi_j] \quad (91)$$

or

$$K_2 - K_2^0 = \sum_{n=-\infty}^{\infty} d(n) \times\cos[l\Phi - (\Omega_i - \Omega_j)t + \phi_i - \phi_j]. \quad (92)$$

It is possible to rewrite the summation in such a way that one of its limits is finite and terms of same magnitude are added together. To do this we notice that if l is odd,

$$\sum_{n=-\infty}^{\infty} d(n) = \sum_{n=(l+1)/2}^{\infty} d(n) + d(l-n), \quad (93)$$

and if l is even,

$$\sum_{n=-\infty}^{\infty} d(n) = d(l/2) + \sum_{n=l/2+1}^{\infty} d(n) + d(l-n) = \sum_{n=l/2}^{\infty}\left(1 - \frac{\delta_{n(l/2)}}{2}\right)[d(n) + d(l-n)], \quad (94)$$

where δ represents the Kronecker delta function.

Evaluation of $d(n) + d(l-n)$ for odd l leads to the expression

$$K_2 = K_2^0 + \frac{A_iA_j}{2} \times\sum_{n=(l+1)/2}^{\infty}\frac{l^2V_nV_{n-l}(n-l)n(\Omega_i + \Omega_j)}{l^2\Omega_i\Omega_j + n(l-n)(\Omega_i - \Omega_j)^2} \times\left(\frac{V'_n}{nV_n} + \frac{V'_{n-l}}{(n-l)V_{n-l}} + \frac{(2n-l)(\Omega_i - \Omega_j)}{l^2\Omega_i\Omega_j + n(l-n)(\Omega_i - \Omega_j)^2}l\Omega'_i\right) \times\cos[l\Phi - (\Omega_i - \Omega_j)t + \phi_i - \phi_j], \quad (95)$$

and for even l:

$$K_2 = K_2^0 + \frac{A_iA_j}{2} \times\sum_{n=l/2}^{\infty}\left(1 - \frac{\delta_{n(l/2)}}{2}\right)\frac{lV_nV_{n-l}(n-l)n}{l^2\Omega_i\Omega_j + n(l-n)(\Omega_i - \Omega_j)^2} \times\left[\left(\frac{V'_n}{nV_n} + \frac{V'_{n-l}}{(n-l)V_{n-l}}\right)(2n-l)(\Omega_i - \Omega_j) + \frac{l^2(\Omega_i^2 + \Omega_j^2) - 2n(l-n)(\Omega_i - \Omega_j)^2}{l^2\Omega_i\Omega_j + n(l-n)(\Omega_i - \Omega_j)^2}l\Omega'_i\right] \times\cos[l\Phi - (\Omega_i - \Omega_j)t + \phi_i - \phi_j]. \quad (96)$$

In both cases, K_2 can be written as

$$K_2 = K_2^0 + K_2^- \cos[l\Phi - (\Omega_i - \Omega_j)t + (\phi_i - \phi_j)]. \quad (97)$$

The resonance described by K_2 has a full width, in the $\dot{\Phi}$ variable, equal to

$$\Gamma_- = 4\epsilon\left(\left|\frac{\partial^2 h_0(J)}{\partial J^2}K_2^-\right|\right)^{1/2}, \quad (98)$$

it will oscillate in the potential well with frequency

$$\Delta\omega = \frac{\Delta\omega_i - \Delta\omega_j}{l} = \frac{\omega_i - \omega_j}{l}, \quad (99)$$

and will radiate at the frequencies

$$\omega = \omega_1 + [n(\omega_i - \omega_j)/l]. \quad (100)$$

If for odd l we take the case of two waves extremely distant from the main carrier, but having a finite frequency separation, we can find an approximate expression for K_2, by taking the limit $\Omega_j \to \infty$ with $\Omega_i - \Omega_j$ finite:

$$K_2 = K_2^0 + (A_iA_j/\Omega_i)S_l(J) \times\cos[l\Phi - (\Omega_i - \Omega_j)t + \phi_i - \phi_j], \quad (101)$$

with

$$S_l(J) = \sum_{n=(l+1)/2}(n-l)V_{n-l}V'_n + nV_nV'_{n-l}. \quad (102)$$

This resonance will have a width

$$\Gamma_- = \frac{4\epsilon}{\sqrt{\Omega_i}}\left(A_iA_j\left|\frac{\partial^2 h_0(J)}{\partial J^2}S_l(J)\right|\right)^{1/2}. \quad (103)$$

Since $\Gamma_- \propto 1/\sqrt{\Omega_i}$, the resonance width may be sizable even when large distances separate the main wave from the perturbation. In particular if a large set of weak and equally spaced waves perturbs the main carrier, the total perturbation will be a function of a large number of those weak waves, and the overall effect, although in practice only a few times larger in magnitude than the one created by a single pair of lines, will be nonlocal, with properties not entirely attributable to any wave pair in particular.

The general expression for K_2 is a sum over n of terms containing as a denominator the expression

$$l^2\Omega_i\Omega_j + n(l-n)(\Omega_i - \Omega_j)^2 = [(l-n)\Omega_i + n\Omega_j] \times [(l-n)\Omega_j + n\Omega_i].$$

If for some n we have either

$$n = n_1 \approx l\Omega_i/(\Omega_i - \Omega_j) \tag{104}$$

or

$$n = n_2 \approx -[l\Omega_j/(\Omega_i - \Omega_j)], \tag{105}$$

the summation can be approximated by the term that has such a value of n alone. Since $n_1 + n_2 = l$, $n_1 \neq n_2$, and $n_{1,2} \geqslant l/2$, at most one value of n will satisfy any of the two equations. Such value if it exists will be $n = \max(n_1, n_2)$, which we will assume for simplicity to be n_1. Because the perturbation expansion converges only if there is no resonance overlap inside the carrier potential well, which causes chaos, and the enhancement of the term we are looking at is caused by the existence of a $|n_1|$-lobed first-order resonance caused by wave i alone, we must have as the closest possible spacing for the two resonances:

$$\frac{\Omega_i - \Omega_j}{l} - \frac{\Omega_i}{n} = 2\sqrt{\epsilon A_i |V_n|/8}. \tag{106}$$

Under those conditions K_2^0 and K_2^- can be rewritten as

$$K_2^0 \approx f_2^0/\sqrt{\epsilon}, \tag{107}$$

$$K_2^- \approx (f_2^-/\sqrt{\epsilon})\cos[l\Phi - (\Omega_i - \Omega_j)t + \phi_i - \phi_j], \tag{108}$$

$$K = h_0 + \epsilon^{3/2} f_2^0 + \epsilon^{3/2} f_2^- \times \cos[l\Phi - (\Omega_i - \Omega_j)t + \phi_i - \phi_j]. \tag{109}$$

This will lead to an enhanced resonance width:

$$\Gamma_- = \epsilon^{3/4}\left(\left|\frac{\partial^2 h_0(J)}{\partial J^2} f_2^-\right|\right)^{1/2}, \tag{110}$$

putting the resonance halfway between a first- and second-order effect in terms of its magnitude.

We note that the width of this type of resonance when created by the processes described in part I of the paper is a totally symmetric function of the wave amplitudes, i.e., the width remains invariant if any permutation is made among the interacting carriers. This is not the case when the same effect is described by a trapped resonance since it is highly convenient that the wave that contains the resonance be the strongest of them all, so that the required value of $|l|$ is minimized and the radiation is as strong as possible.

2. Frequency addition

If we take two distinct waves with frequencies Ω_i and Ω_j, the term symmetrical under permutation of the indices i and j gives

$$4(K_2 - K_2^0) = \sum_{m+n=1} (\alpha_{ijmn} + \alpha_{jimn}) \times \cos[l\Phi - (\Omega_i + \Omega_j)t + \phi_i + \phi_j], \tag{111}$$

with

$$\alpha_{ijmn} = A_j(ma'_{in} V_m - na_{in} V'_m). \tag{112}$$

Calculation of the α's leads to the following expression for K_2:

$$K_2 - K_2^0 = \frac{A_i A_j}{4} \sum_{n=-\infty}^{\infty} \frac{l V_n V_{l-n}(l-n)n}{l^2\Omega_i\Omega_j - n(l-n)(\Omega_i + \Omega_j)^2} \times \left[(2n-l)(\Omega_i + \Omega_j)\left(\frac{V'_n}{nV_n} - \frac{V'_{l-n}}{(l-n)V_{l-n}}\right) - \frac{l^2(\Omega_i^2 + \Omega_j^2) - 2n(l-n)(\Omega_i + \Omega_j)^2}{l^2\Omega_i\Omega_j - n(l-n)(\Omega_i + \Omega_j)^2} l\Omega'_i\right] \times \cos[l\Phi - (\Omega_i + \Omega_j)t + \phi_i + \phi_j] \tag{113}$$

or

$$K_2 - K_2^0 = \sum_{n=-\infty}^{\infty} c(n)\cos[l\Phi - (\Omega_i + \Omega_j)t + \phi_i + \phi_j]. \tag{114}$$

Since $c(n) = c(l-n)$, we are led to the following simplifications:

For odd l,

$$K_2 = K_2^0 + \frac{1}{2}\sum_{n=(l+1)/2}^{\infty} c(n) \times \cos[l\Phi - (\Omega_i + \Omega_j)t + \phi_i + \phi_j]. \tag{115}$$

For even l,

$$K_2 = K_2^0 + \frac{1}{4}\sum_{n=l/2} (2 - \delta_{n(l/2)})c(n) \times \cos[l\Phi - (\Omega_i + \Omega_j)t + \phi_i + \phi_j]. \tag{116}$$

In both cases, K_2 can be written as

$$K_2 = K_2^0 + K_2^+ \cos[l\Phi - (\Omega_i + \Omega_j)t + (\phi_i + \phi_j)]. \tag{117}$$

The resonance described by K_2 has a full width, in the $\dot{\Phi}$ variable, equal to

$$\Gamma_+ = 4\epsilon\left(\left|\frac{\partial^2 h_0(J)}{\partial J^2} K_2^+\right|\right)^{1/2}, \tag{118}$$

it will oscillate in the potential well with frequency

$$\Delta\omega = (\Delta\omega_i + \Delta\omega_j)/l, \tag{119}$$

and will radiate at the frequencies

$$\omega = \omega_1 + [n(\Delta\omega_i + \Delta\omega_j)/l]. \tag{120}$$

An example of this type of resonance is shown in Fig. 5 for $l = -2$, $\Omega_i = 7\Omega_0$, and $\Omega_j = -9\Omega_0$, with $\Omega_0 \approx \Omega_{t1}$. In this example it can be seen that the main wave trapping frequency imposes no constraints on the required perturbing carrier frequency separations or on the bandwidth and line separations of the radiated spectrum. The perturbing carriers are positioned several times Ω_{t1} away from the main

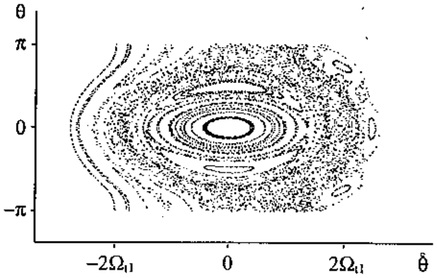

FIG. 5. Frequency addition intermodulation resonance for $l = -2$, $\Omega_i = 7\Omega_0$, and $\Omega_j = -9\Omega_0$, with $\Omega_0 = 6/7\Omega_{t1}$. The resonance consists of the two crescent shapes symmetrically placed relative to the $\theta = 0$ axis, and centered at $\dot\theta = 0$. As time flows, they will turn with frequency $\Delta\omega = (\Omega_i + \Omega_j)/l\,[v_g/(v_\parallel + v_g)]$. The three lobes at right, located at $\dot\theta = 2\Omega_{t1}$ and $\theta = 0, \pm 2\pi/3$, represent an external 1/3-harmonic resonance created by wave i and the main wave. At the left the external resonance for the interaction $\omega = \omega_i + \omega_j - \omega_1$ ($\dot\theta = -2\Omega_0$ and $\theta = \pi$) can also be seen. A sizeable chaotic band is seen circling the main resonance. Its presence will decrease main wave growth.

wave potential, and the resonance produces a spectrum with main lines separated by $2\Omega_0 > \Omega_{t1}$.

Similarly to the previous subsection, due to the poles in its denominators, K_2 will be enhanced if

$$n = n_{1,2} \approx l\Omega_{i,j}/(\Omega_i + \Omega_j), \qquad (121)$$

Under those conditions K_2^0 and K_2^+ can be written as

$$K_2^0 \approx f_2^0/\sqrt{\epsilon}, \qquad (122)$$

$$K_2^+ \approx (f_2^+/\sqrt{\epsilon})\cos[l\Phi - (\Omega_i + \Omega_j)t + \phi_i + \phi_j], \qquad (123)$$

or

$$K = h_0 + \epsilon^{3/2} f_2^0 + \epsilon^{3/2} f_2^+ \times\cos[l\Phi - (\Omega_i + \Omega_j)t + \phi_i + \phi_j], \qquad (124)$$

which also leads to a larger resonance width:

$$\Gamma_- = \epsilon^{3/4}\left(\left|\frac{\partial^2 h_0(J)}{\partial J^2} f_2^+\right|\right)^{1/2}. \qquad (125)$$

C. Two wave effects: Second harmonic

If in the expression for $\{W_1, h_1\}$, we select the terms symmetrical under permutation of i and j, but assume $i = j$, we get a new type of resonance. Since we do not want this resonance to fall on top of any already present first-order resonances, we choose $m + n = l$ to be an odd number. The expression for K_2 can then be either directly calculated or obtained from the frequency addition resonance by putting $i = j$ and dividing all terms by two. We then get

$$K_2 = K_2^0 + \frac{A_i^2}{2}\sum_{n=(l+1)/2}^{\infty}\frac{lV_n V_{l-n}(l-n)n}{(2n-l)\Omega_i} \times\left(\frac{V_n'}{nV_n} - \frac{V_{l-n}'}{(l-n)V_{l-n}} - \frac{l\Omega_i'}{(2n-l)\Omega_i}\right) \times\cos(l\Phi - 2\Omega_i t + 2\phi_i), \qquad (126)$$

which can be written as

$$K_2 = K_2^0 + K_2^2\cos(l\Phi - 2\Omega_i t + 2\phi_i). \qquad (127)$$

The resonance described by K_2 has a full width, in the $\dot\Phi$ variable, equal to

$$\Gamma_2 = 4\epsilon\left(\left|\frac{\partial^2 h_0(J)}{\partial J^2} K_2^2\right|\right)^{1/2}; \qquad (128)$$

it will oscillate in the potential well with frequency

$$\Delta\omega = 2\Delta\omega_i/l \quad (\text{odd } l) \qquad (129)$$

and will radiate at the frequencies

$$\omega = \omega_1 + (2n\Delta\omega_i/l) \quad (\text{odd } l). \qquad (130)$$

This is a second harmonic resonance that will produce most of its radiation at even multiples of the frequency separation, $\Delta\omega_i$. There will also be some radiation at subharmonic frequencies, adding to the spectrum weaker lines separated by multiples of $2\Delta\omega_i/l$.

VIII. INTERFERENCE EFFECTS

In the case of first-order resonances, if we happen to have two waves, l and j, such that $\Omega_l = -\Omega_j$, the same value of $\dot\Phi$ will be simultaneously resonant with both. We are faced with the case of resonance interference. For a first-order Hamiltonian we must then choose

$$K_1 = -A_l V_{-m}(J)\cos(-m\Phi - \Omega_l t + \phi_l) - A_j V_m(J) \times\cos(m\Phi - \Omega_j t + \phi_j), \qquad (131)$$

which is equal to

$$K_1 = -A_l(-1)^m V_m(J)\cos(m\Phi - \Omega_j t - \phi_l) - A_j V_m(J) \times\cos(m\Phi - \Omega_j t + \phi_j), \qquad (132)$$

and can be rewritten as

$$K_1 = r\cos(m\Phi - \Omega_j t - \Phi_0), \qquad (133)$$

with

$$\tan\Phi_0 = \frac{A_l(-1)^m\sin\phi_l - A_j\sin\phi_j}{A_l(-1)^{m)}\cos\phi_l + A_j\cos\phi_j} \qquad (134)$$

and

$$r^2 = A_l^2 + A_j^2 + 2(-1)^m A_j A_l\cos(\phi_j + \phi_l). \qquad (135)$$

If $\phi_j + \phi_l = \pi$,

$$r = |V_m|\,|A_j(-1)^m - A_l|, \qquad (136)$$

which is minimized if m is even, and zero if additionally $A_j = A_l$. In this case, all even m, first-order radiated subharmonics will have zero amplitude, and odd ones will be reinforced.

If $\phi_j + \phi_l = 0$,

$$r = |V_m|\,|A_j(-1)^m + A_l|, \qquad (137)$$

which is minimized for m odd, and zero if $A_j = A_l$. If this happens, all even m first-order radiated subharmonics will be reinforced, and the odd ones will be weakened. The odd ones will not be completely cancelled because it is always possible to get radiation at their frequencies from an even resonance with an appropriate m.

Those interference effects are a simple consequence of the well-defined parity properties of the Hamiltonians generated in the wave-wave interaction process. Such reflection properties will also be present in higher-order terms. For instance, it is possible to show that for all of the already calculated second-order terms, the parity of the Hamiltonian is equal to $(-1)^l$. As a consequence, interference effects, similar in nature to the ones above described, should also occur among them.

IX. HIGHER-ORDER EFFECTS

From the examples seen in the previous sections and from the way each term is obtained, it is possible to extrapolate the possible frequencies for effects of order n:

$$\Delta\omega = \sum_{i=2}^{N} n_i \Delta\omega_i / l, \tag{138}$$

with $\Sigma_{i=2}^{N} |n_i| = n$. It is clear that terms of order n contain at most $n+1$ interacting waves.

Two given waves can generate a n order harmonic by two different processes: (1) First harmonic radiation by a n-order harmonic resonance and (2) high-order harmonic radiation from a low-order resonance.

The n-order resonance widths are of the order $\epsilon^{n/2}$. The amplitude of their first harmonics should therefore go down exponentially with n. High-order harmonics coming from a low-order resonance have amplitudes proportional to the Bessel functions J_n which decrease in value, as a function of n, faster than an exponential, and can therefore be neglected. We are led then to the conclusion that harmonic sidebands created by two waves through trapping effects should have an exponential amplitude slope as a first approximation, as is indeed borne out by the data.

X. CONCLUSIONS

As main results from trapping effects we can quote the following:

(1) Because their frequencies are independent of $v_\perp$, and because if $|n|$ is large enough a resonance will fit in any wave potential, sidebands have narrow linewidths and well-defined frequencies, and can be produced independently of carrier amplitude values or variations. However, weaker waves will have a spectrum more finely divided by subharmonics because higher values of $|n|$ will be involved. Integration over different pitch angles will not smear out the resonances.

(2) Electrons interact with Doppler-shifted wave separations which may be larger than the nominal frequency separations by a factor of 3 or more. The resulting effect can be understood as either a decrease in the time constants of the system or a decrease in the interacting wave amplitudes.

(3) Since around the equator electrons can always be temporarily trapped, i.e., they will describe part of a closed orbit if put inside the main wave potential well, electron streams can always exist giving rise to trapping effects with arbitrarily weak fields, independently of inhomogeneity effects.

(4) Due to the frequency modulation process, sidebands are created in symmetric pairs. Line growth can change their amplitudes and symmetry appreciably.

(5) Trapping predicts an approximately exponential fall-off for the harmonic spectrum, line-line interference effects, and a slow fall-off with distance (in frequency space) of the interaction strength for some resonance types. This last point indicates that a comb of weak, equally separated lines can appreciably affect sideband formation if the number of lines is large enough (the effect is divergent for an infinite set of lines).

(6) Although line amplitudes are different, sidebands due to trapping effects are created at the same frequencies as the ones due to external resonances:

$$\omega = \sum_{i=1}^{N} n_i \omega_i \Big/ \sum_{i=1}^{N} n_i\,. \tag{31}$$

(7) Sidebands due to trapping also need a nonzero gradient in $v_\parallel$ to be created.

(8) Individual electron oscillations and the continuous spectrum of their frequencies ("trapping" oscillation frequencies) do not directly affect the observed sideband spectra. Those oscillations are convective, occurring as the electron, as part of a stream, moves along the interaction region, and may not even exist for a full cycle. Radiated frequencies are associated with oscillation frequencies of the streams to which the electrons belong. Those oscillation frequencies are only indirectly related to the trapping frequencies, and the frequencies they radiate may be almost an order of magnitude away from available trapping frequencies. The total emitted radiation will be a combined result from all such stream oscillations, and its spectrum will reflect the frequency structure of the externally accelerating forces, and not of the overall electron population.

(9) Chaos, always present when internal resonances are formed, is an important factor in the formation of the final radiation spectra, contributing to saturation effects.

(10) Trapped resonances are more protected from inhomogeneity effects than external resonances and should produce radiation over a longer length of the interaction region around the equator.

ACKNOWLEDGMENTS

The author thanks his StarLab colleagues for helpful discussions. This work was sponsored by the Division of Polar Programs of the National Science Foundation under Grant No. DPP-86-13783.

[1]G. R. Smith and N. R. Pereira, Phys. Fluids **21**, 2253 (1978).