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The Nonlinear Gyroresonance Interaction Between Energetic Electrons and Coherent VLF Waves Propagating at an Arbitrary Angle With Respect to the Earth's Magnetic Field

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A theory is presented of the nonlinear gyroresonance interaction that takes place in the magnetosphere between energetic electrons and coherent VLF waves propagating in the whistler mode at an arbitrary angle ψ with respect to the earth's magnetic field $\mathbf{B}_0$. In particular, we examine the phase trapping (PT) mechanism believed by some to be responsible for the generation of VLF emissions. The phase considered in this study is the phase γ between the right-hand circularly polarized component of the wave magnetic field perpendicular to $\mathbf{B}_0$ and the component of the energetic particle velocity vector perpendicular to $\mathbf{B}_0$. This model is an extension of one developed in earlier work [*Bell*, 1965; *Dysthe*, 1971; *Nunn*, 1974] involving the special case where $\psi = 0$. The extended theory predicts that for any finite value of ψ there is a range of resonant particle pitch angle α for which γ is not bounded within the range $0 \leq |\gamma| < \pi$, and thus PT in the usual sense is not possible. However, PT in an average sense can still exist and long term energy transfer between the gyroresonant electrons and the wave can still take place. It is found that for given values of ψ and wave frequency, the trapping frequency ω_τ has a countably infinite set of zeros when plotted as a function of α. Regions between zeros alternate between regions of normal PT and regions of anomalous PT. The average phase angle of anomalously phase trapped electrons differs by approximately 180° from the average phase of the normal phase trapped electrons. However, the average stimulated radiation from the two groups of particles tends to add in phase. Resonant particles for which $\omega_\tau \Rightarrow 0$ are not phase trapped. Near the geomagnetic equatorial plane, the threshold value of wave amplitude necessary to produce PT is directly proportional to the gradient of ψ along $\mathbf{B}_0$. A simple model predicts threshold values greater than $10\, m\gamma$ for $\psi > 37°$. For interactions far from the magnetic equator, ψ variations appear to be less important than gradients in $\mathbf{B}_0$ and in this case we calculate the wave power increase necessary to produce the same PT efficiency for typical nonducted waves as that appropriate to ducted waves. The component of the wave electric field parallel to $\mathbf{B}_0$ generally enhances the strength of the PT process and tends to dominate the process for moderate to high ψ at midfrequency. It is concluded that near the magnetic equatorial plane gradients of ψ may play a very important part in the PT process for nonducted waves. Predictions of a higher threshold value for PT for nonducted waves generally agree with experimental data concerning VLF emission triggering by nonducted waves.

1. INTRODUCTION

This paper presents a study of the nonlinear "phase" trapping interaction that takes place between coherent VLF waves and energetic gyroresonant electrons in the magnetosphere. The phase concerned in this study is the phase γ between $\mathbf{B}_R$ and $\mathbf{v}_\perp$, where $\mathbf{B}_R$ is the right-hand circularly polarized component of the wave magnetic field in the plane perpendicular to the earth's magnetic field $\mathbf{B}_0$ and $\mathbf{v}_\perp$ is the component of the velocity of the gyroresonant particle perpendicular to $\mathbf{B}_0$. When the phase angle is bounded, the particle is said to be "phase trapped." In the case of waves propagating parallel to $\mathbf{B}_0$, it has been shown that phase trapping results in long-term energy transfer between the wave and the particle [*Bell*, 1965; *Dysthe*, 1971; *Nunn*, 1974]. Furthermore, in the case of parallel propagation, the phase trapped gyroresonant electrons constitute a coherent transverse current that can radiate and alter the wave structure, affecting any subsequent phase trapping of new electrons. This results in a feedback process that has been advanced as the explanation of artificially stimulated VLF emissions [*Helliwell and Crystal*, 1973].

Our work differs from earlier work on the phase trapping process in that we consider the case in which the wave normal angle of the VLF wave can have an arbitrary angle with respect to $\mathbf{B}_0$. Previous work [*Bell*, 1965; *Dysthe*, 1971; *Matsumoto and Kimura*, 1971; *Nunn*, 1971, 1974; *Palmadesso and Schmidt*, 1971, 1972; *Ashour-Abdalla*, 1972; *Bud'ko et al.*, 1972; *Helliwell and Crystal*, 1973; *Roux and Pellat*, 1976, 1978; *Karpman et al*, 1974a,b, 1975; *Vomvoridis and Denavit*, 1979; *Matsumoto et al.*, 1980; *Bell and Inan*, 1981; *Matsumoto and Omura*, 1981; *Omura and Matsumoto*, 1982] has centered almost exclusively on the case in which the VLF waves propagate parallel to $\mathbf{B}_0$ within a whistler mode duct [*Helliwell*, 1965] with $\psi = 0$. That assumption was reasonable within the context of early VLF wave injection experiments [*Helliwell and Katsufrakis*, 1974; *McPherson et al.*, 1974; *Koons et al.*, 1976; *Dowden et al.*, 1978] since all observations were made on the ground where only ducted waves could be observed. However, recent satellite observations of VLF emission triggering by nonducted waves [*Inan et al.*, 1977; *Bell et al.*, 1981; *Helliwell and Inan*, 1982; *Bell et al.*, 1983; *Kimura et al.*, 1983] show the need for a more general theory of the phase trapping process.

In our work we show that an analog of the phase trap-

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ping process exists for the case $\psi \neq 0$ and that this process can also result in long-term energy transfer between the wave and the resonant particles. However, we show that many important differences exist between the cases of $\psi = 0$ and $\psi \neq 0$. In particular, there is a unique feature of the interaction between nonducted VLF waves and energetic electrons in the magnetosphere which has no analog in the theory of ducted interactions and which is important to note here. This feature concerns the variation of wave normal angle ψ.

Let us consider the simple two-dimensional case in which VLF waves injected by a ground-based transmitter into a meridional plane arrive at some constant altitude reference level, say, 1000 km altitude, above which the theory of ray optics holds. We also assume that both the wave normal direction and the phase of the wave structure at the reference level is known as a function of latitude. With this information we can then calculate the phase of the wave structure at higher altitudes using ray tracing equations and the WKB approximation for the wave phase

$$\Phi(\mathbf{r}) = \int \omega dt - \int \mathbf{k} \cdot dr \qquad (1)$$

where the integral is to be evaluated along the ray direction, where $\mathbf{k}$ is the wave vector, and $\mathbf{r}$ is the position vector along the ray path. In Figure 1 we show schematically the ray paths originating at two closely spaced values of input latitude. These ray paths are shown intersecting a particular magnetic field line. In general, these rays will possess different values of wave normal direction at their intersections even if their initial wave normal directions were the same. This situation arises because each ray traverses slightly different portions of the magnetosphere where the magnitudes and spatial gradients of the cold plasma density and the ambient magnetic field are slightly different. Furthermore, the rays cross the magnetic field line at slightly different altitudes because of the finite dip angle of the magnetic field.

As a result of these differences, an energetic electron moving along a magnetic field line illuminated by a nonducted wave will encounter a wave structure in which the local wave normal direction is a function of position along $\mathbf{B}_0$.

The variation of ψ along $\mathbf{B}_0$ can have an important effect upon gyroresonance interactions between energetic electrons and nonducted VLF waves in the magnetosphere since the local resonance velocity is a function of ψ. Thus variations in ψ can serve to alter the duration and efficiency of gyroresonance interactions. The effects of gradients in ψ are discussed in detail in sections 2 and 3.

The contributions of the present work include the following. We examine the equations of motion of an energetic electron moving along $\mathbf{B}_0$ approximately in gyroresonance with a nonducted coherent VLF wave and establish the conditions under which phase trapping can exist. We show that phase trapping in the usual sense is not possible in general but that phase trapping in an average sense can still occur and that this average phase trapping is associated with long-term energy transfer between wave and particles. We show that for a given value of wave normal angle, and normalized wave frequency, the trapping frequency ω_T has a countably infinite set of zeros when plotted as a function of the resonant particle pitch angle. Regions between zeros alternate between regions of normal phase trapping and regions of "anomalous" phase trapping (see definition on page 14). The average phase angle of anomalously phase trapped electrons differs by approximately 180° from the average phase of the normally phase trapped electrons. However, the average stimulated radiation from the two groups of particles tends to add in phase. Resonant particles for which $\omega_T \Rightarrow 0$ are not phase trapped and the stimulated radiation from these particles tends to cancel that of the phase trapped particles.

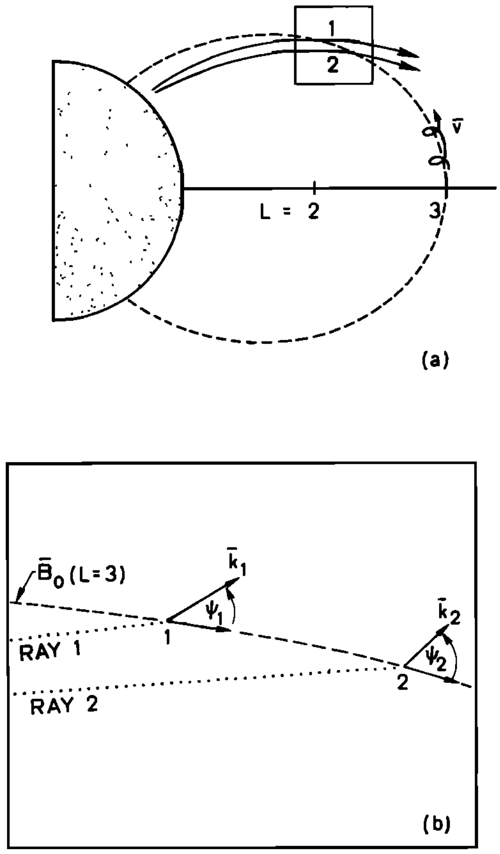

Fig. 1. Schematic representation of the illumination of a magnetic field line by a nonducted VLF wave from a ground-based transmitter. (a) Ray paths of wave energy injected into the ionosphere at different latitudes generally intersect a particular magnetic field line at different points. Details of intersection points in inset are shown in (b). (b) Wave normal angles of the wave at the two intersection points are generally different due to differences in local conditons along the ray path and to differences in optical path length. Energetic particles moving along the magnetic field line consequently encounter a wave structure in which the wave normal direction is a function of distance along the fieldline.

We develop explicit expressions that show how gradients in ψ and other parameters affect the phase trapping process and show that near the geomagnetic equatorial plane, the threshold value of wave amplitude necessary to produce phase trapping is directly proportional to the gradient of ψ along $\mathbf{B}_0$. A simple model for the variation of ψ leads to threshold values $> 10m\gamma$ for $\psi > 37°$ when $L \sim 4$.

It is shown that gradients in ψ can be neglected only at latitudes far from the magnetic equator or in regions where the cold plasma density decreases rapidly for increasing L value. In these cases it is shown that at low normalized frequencies ($\omega/\omega_H \sim 0.1$, where ω_H = electron gyrofrequency) wave power increases of 5 dB or more are necessary to produce the same phase trapping efficiency for typical nonducted waves as that appropriate to ducted waves of the same frequency. However, at mid-frequency ($\omega \sim 0.5\omega_H$) the power increase necessary to maintain constant phase trapping efficiency at typical wave normal angles is generally <3 dB. We show that the component of the wave electric field parallel to earth's magnetic field generally acts to enhance the strength of the phase trapping process and tends to dominate the process for moderate to high wave normal angles when the wave frequency lies in the range $0.5 \leq \omega/\omega_H < 1$.

We develop time averaged (over one gyroperiod) equations describing the energetic electron motion and show how the average pitch angle change of the resonant electrons is related to the trapping frequency and to the average phase angle $\underline{\gamma}$. We show how pitch angle changes can lead to detrapping of resonant electrons in a way which is a natural consequence of the particle dynamics.

We derive the form of the phase trapping equations for gyroresonance interactions of arbitrary harmonic number and show that higher order phase trapping is much less efficient than the phase trapping associated with the fundamental gyroresonance interaction.

We conclude that gradients in ψ may play a very important part in the phase trapping process for nonducted waves.

2. Theory

We consider the motion of a test electron mirroring in the earth's magnetic field under the influence of a whistler mode wave of slowly varying amplitude, frequency, and wave number which is propagating in the $x-z$ plane at an angle ψ with respect to the negative z axis (see Figure 2a). The magnetic and electric fields of this wave can be written in the form:

$$\mathbf{B}_w = \mathbf{e_x} B_x^w \cos\Phi + \mathbf{e_y} B_y^w \sin\Phi - \mathbf{e_z} B_z^w \cos\Phi$$

$$\mathbf{E}_w = -\mathbf{e_x} E_x^w \sin\Phi + \mathbf{e_y} E_y^w \cos\Phi - \mathbf{e_z} E_z^w \sin\Phi \qquad (2)$$

where the wave phase Φ has the representation given in (1), and where $\mathbf{e_i} (i = x, y, z)$ are unit vectors along the x, y, and z axes, respectively.

We assume that locally the earth's magnetic field $\mathbf{B}_o$ is directed along the positive z axis of our right-handed coordinate system and that the propagation characteristics of the whistler mode wave are governed mainly by the cold plasma component of the magnetospheric plasma. In this case the wave electric field components and the polarization ratios ($\mathbf{B}_y^w/\mathbf{B}_x^w$, etc.) can be derived from Maxwell's equations:

$$\mathbf{k} \times E_w = -\omega \mathbf{B}_w$$

$$\mathbf{k} \times B_w = c^{-2} \mathbf{K} \cdot E_w \qquad (3)$$

where $\mathbf{K}$ is the dielectric tensor as defined by *Stix* [1962] and c is the velocity of light.

2.1. Equations of Motion

The particle motion can be determined by the Lorentz force law:

$$\ddot{\mathbf{r}} = -\frac{e}{m}\left[\mathbf{E}_w + \dot{\mathbf{r}} \times [\mathbf{B}_w + \mathbf{B}_0(\mathbf{r})]\right] \qquad (4)$$

where $\mathbf{E}_w$ and $\mathbf{B}_w$ are given by (1), $\mathbf{r}$ is the electron position vector, e is the magnitude of the electron charge, and m is the electron mass.

We assume that $\mathbf{B}_0$ is slowly varying and locally parallel to the z axis. However, this last condition cannot be strictly true if $\mathbf{B}_0$ varies with z, since then the relation $\nabla \cdot \mathbf{B}_0 = 0$ requires that $\mathbf{B}_0$ have a component in the $x-y$ plane, $\mathbf{B}_{0\perp}$. It is the effect of $\mathbf{B}_{0\perp}$ that is responsible for the "mirror" force $\mathbf{F}_m = q[\mathbf{v}_\perp \times \mathbf{B}_{0\perp}]$, which traps energetic particles in the earth's magnetic field [*Alfven*, 1950].

The mirror force considered in past work [e.g., *Dysthe*, 1971; *Inan et al.*, 1978; *Bell and Inan*, 1981] has been the temporal average of $\mathbf{F}_m$ over one gyroperiod. In the present work we will be concerned initially with variations in $\mathbf{v}$ and $\mathbf{r}$ that take place during a single gyroperiod. Consequently, we will initially consider the more general form of $\mathbf{F_m}$ in our equations.

In what follows we assume that $\mathbf{B}_0$ is exactly parallel to the z axis at all positions $x = y = 0$ and evaluate $\mathbf{B}_{0\perp}$ by means of a Taylor expansion in x and y. For a dipole model of $\mathbf{B}_0$, it can be shown that:

$$\mathbf{B}_{0\perp} \approx -(\mathbf{e_x} \cos\xi_0 + \mathbf{e_y} \sin\xi_0)(x \cos\xi_0 + y \sin\xi_0)\frac{\partial B_{0z}}{\partial z}$$

where ξ_0 is the angle between the magnetic meridional plane and the $x-z$ plane. The gyroradius of the resonant energetic particles we will consider is generally <1 km near the magnetic equatorial plane. In a dipole model for $L = 4$ the fractional change in $\mathbf{B}_{0z}$ over this distance is approximately 10^{-4}, thus we assume below that $\mathbf{B}_{0z}(x, y, z) = \mathbf{B}_{0z}(0, 0, z) \equiv \mathbf{B}_0(z)$. For similar reasons we also assume that $N_0 = N_0(z)$, where N_0 is the cold plasma density.

It is convenient to decompose the wave magnetic field in the $x-y$ plane into two circularly polarized components with opposite senses of rotation, i.e.,

$$\mathbf{B}_R = \frac{B_x^w + B_y^w}{2}[\mathbf{e}_x \cos\Phi + \mathbf{e}_y \sin\Phi]$$

$$\mathbf{B}_L = \frac{B_x^w - B_y^w}{2}[\mathbf{e}_x \cos\Phi - \mathbf{e}_y \sin\Phi] \qquad (5)$$

Figure 2*b* shows the geometry of this decomposition. The circularly polarized field component $\mathbf{B_R}$ is oriented at the angle Φ with respect to the x axis while the component $\mathbf{B}_L$ is oriented at the angle $-\Phi$ with respect to the x axis. At a fixed point in space, $\mathbf{B_R}$ rotates counterclockwise about $\mathbf{B}_0$ with the angular velocity ω, while $\mathbf{B}_L$ rotates clockwise with the angular velocity ω. Consequently, $\mathbf{B}_R$ rotates in the same sense about $\mathbf{B}_0$ as would an electron.

We now write the equations of motion of the particle in a velocity space coordinate system involving the velocity components both parallel (v_z) and perpendicular ($\mathbf{v}_\perp$) to $\mathbf{B}_0$ and involving the angle γ between $\mathbf{v}_\perp$ and $\mathbf{B_R}$. From Figure 2*c* it can be seen that

$$\dot{\gamma} = \dot{\theta} - \dot{\Phi} \qquad (6a)$$

where θ is the angle between $\mathbf{v}_\perp$ and the x axis and is defined by the relation:

$$\theta = \tan^{-1}(v_y/v_x) \qquad (6b)$$

Equations (4), (5), and (6) yield the set:

$$\dot{\gamma} = \omega_H - \omega - k_z v_z - k_x v_x - A_1 \qquad (7)$$

$$\dot{v}_z = \omega_1 v_\perp \sin\gamma + \omega_2 v_\perp \sin(\gamma + 2\Phi) + \frac{e}{m} E_z^w \sin\Phi - A_2 \quad (8)$$

$$\dot{v}_\perp = -\omega_1 (v_z + R_1)\sin\gamma - \omega_2 (v_z - R_2)\sin(\gamma + 2\Phi) + A_3 \quad (9)$$

where $v_x = v_\perp \cos(\gamma + \Phi)$, where $\gamma + \Phi \equiv \theta$, and where

$$A_1 \equiv \frac{e}{m} v_z (\mathbf{v}_\perp \cdot \mathbf{B}_{0\perp}) v_\perp^{-2}$$

$$A_2 \equiv \frac{e}{m} [\mathbf{v}_\perp \times \mathbf{B}_{0\perp}] \cdot \mathbf{e}_z$$

$$A_3 \equiv \frac{e}{m} v_z v_\perp^{-1} (\mathbf{v}_\perp \times \mathbf{B}_{0\perp}) \cdot \mathbf{e}_z \quad (10)$$

where $\omega_H \equiv e/m(\mathbf{B}_0(z))$ and $\omega_{1,2} \equiv e/m(\mathbf{B}_x^w \pm \mathbf{B}_y^w)/2$ and $R_{1,2} = [\mathbf{E}_x^w \pm \mathbf{E}_y^w]/[\mathbf{B}_x^w \pm \mathbf{B}_y^w]$.

In deriving (7) we have neglected terms of the order B_w/B_0, following earlier work [*Dysthe*, 1971; *Inan et al.*, 1978; *Bell and Inan*, 1981]. This approximation is justified since in practice $B_w/B_0 \sim 10^{-4} - 10^{-5}$ and we keep terms only of order $(B_w/B_0)^{1/2}$ or larger.

The terms $A_i (i = 1, 2, 3)$ arise from the instantaneous "mirror" forces exerted on the energetic particle as a result of the inhomogeneity of the ambient magnetic field. Averaging these terms over one gyroperiod leads to the relations:

$$\underline{A}_1 = 0$$

$$\underline{A}_2 = \frac{v_\perp^2}{2\omega_H} \frac{\partial \omega_H}{\partial z}$$

$$\underline{A}_3 = \frac{v_\perp v_z}{2\omega_H} \frac{\partial \omega_H}{\partial z} \quad (11)$$

where the bar under the A_i indicates a time averaged quantity.

It can be shown that $\underline{A}_{2,3}$ are the usual acceleration terms due to the average mirror forces in the inhomogeneous field (e.g., see equation (2) of *Inan et al.* [1978]). For a dipole field it can be shown that the maximum value of A_1 is of the order $10^{-4}\omega_H$. Consequently this term will henceforth be neglected in (7).

Equations (7)-(10) apply to interaction between energetic electrons and nonducted whistler mode waves of arbitrary wave normal angle. For the case of parallel propagating ducted waves, $\psi \Rightarrow 0$, $k_z \Rightarrow k$, $k_x \Rightarrow 0$, $B_x^w \Rightarrow B_y^w$, $\omega_2 \Rightarrow 0$, and $E_z^w \Rightarrow 0$. In this case if we use the average values of the A_i, the above equations reduce to those used by earlier workers to study interactions involving ducted waves [e.g., see *Bell*, 1965; *Dysthe*, 1971; *Inan et al.*, 1978; *Bell and Inan*, 1981].

Equation (7) differs from the analogous equation appropriate to the case of ducted propagation only in the presence of the term $k_x v_x$ on the right-hand side of (7). It is this term that is responsible for most of the unusual features of the phase trapping of energetic particles by nonducted waves. It arises because the motion of the energetic particle parallel to the x axis carries it across the planes of constant phase of the nonducted waves. This situation is shown schematically in Figure 2a. As a result of this motion the energetic particle experiences the wave as a frequency-modulated wave whose modulation index is proportional to the particle gyroradius.

In analyzing the above equations, we wish to determine the conditions under which phase trapping of cyclotron resonant electrons will take place in a nonducted whistler mode wave. In the case of parallel propagation, a necessary condition for phase trapping of a resonant electron is that over a time period of the order of the trapping period, $\tau = 2\pi/\sqrt{\omega_1 k_z v_\perp}$, the condition holds:

$$0 \leq |\gamma| < \pi \quad (12)$$

However, a condition of this kind cannot hold in general for nonducted waves because of the wave phase changes that occur as the resonant electron moves parallel to the x axis. To see this, first assume that a particle is in gyroresonance with the wave at some given time, satisfying the first harmonic gyroresonance condition:

$$v_z \approx v_R = \frac{\omega_H - \omega}{k_z} \quad (13)$$

Over a few gyroperiods (13) will hold, and (7) becomes:

$$\gamma \approx -k_x v_x \approx -k_x v_\perp \cos(\omega_H t + \theta_0) \quad (14)$$

where for v_x we have used the unperturbed value, and where θ_0 is the initial angle with respect to the x axis.

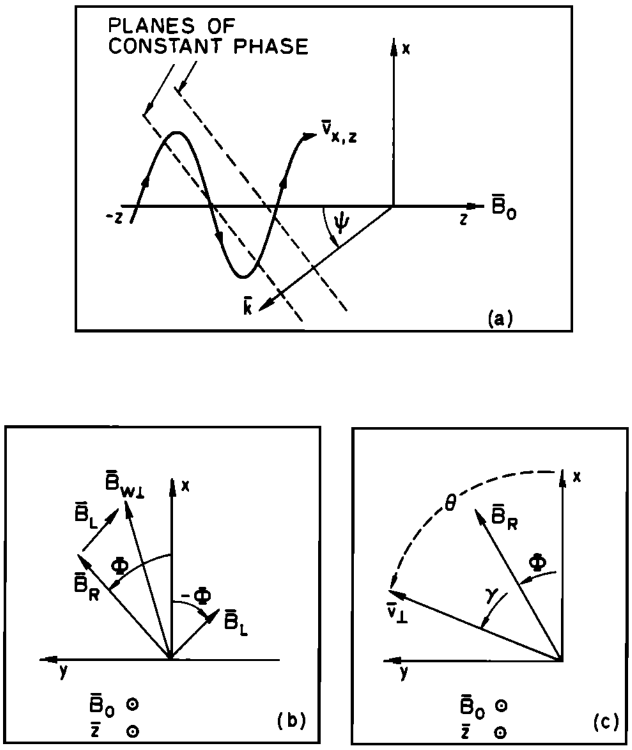

Fig. 2. (a) Geometric schematics of interaction between energetic particles and nonducted coherent wave propagating with a wave normal angle of ψ with respect to the negative z axis, and with the wave vector k lying in the x-z plane. The earth's magnetic field $\mathbf{B}_o$ is assumed to be locally parallel to the z axis. Planes of constant phase of the wave are perpendicular to k and are shown in dashed line. The gyromotion of the particle perpendicular to $\mathbf{B}_o$ carries it across the planes of constant phase, producing an effective frequency modulation in the fields experienced by the particle. (b) Decomposition of the perpendicular wave magnetic field $\mathbf{B}_{w\perp}$ (perpendicular to $\mathbf{B}_o$), into two circularly polarized components $\mathbf{B}_R$ and $\mathbf{B}_L$. $\mathbf{B}_R$ and $\mathbf{B}_L$ rotate in opposite senses, with $\mathbf{B}_R$ rotating in the same sense as the energetic electrons. The z axis and $\mathbf{B}_o$ are both perpendicular to the x-y plane. (c) The quantity γ is the angle between $\mathbf{B}_R$ and $\mathbf{v}_\perp$, where $\mathbf{v}_\perp$ is the component of the energetic electron velocity vector which is perpendicular to $\mathbf{B}_o$. The angle $\theta = \gamma + \Phi$ represents the angle between $\mathbf{v}_\perp$ and the x axis.

Integrating (14) with respect to time, we obtain

$$\gamma \approx -\beta \sin(\omega_H t + \theta_0) + C_0 \quad (15)$$

where C_0 is an arbitrary constant of integration and

$$\beta \equiv \frac{k_x v_\perp}{\omega_H} \equiv (\frac{\omega_H - \omega}{\omega_H}) \tan\psi \tan\alpha \quad (16)$$

The last identity in (16) follows because of (13).

It is clear from (15) that the usual concepts of phase trapping will not apply for $\beta > \pi$ since (12) will not hold. However, phase trapping in an average sense can still exist if the value of γ averaged over one gyroperiod $\underline{\gamma}$ satisfies (12). To show this, we first take the time derivative of (7) and combine this with (8) to obtain an equation for γ:

$$\ddot{\gamma} + \omega_{\tau 0}^2[\sin\gamma + \alpha_1 \sin(\gamma + 2\Phi) + \alpha_2 \sin\Phi] = g(\mathbf{r}, t) - \ddot{\lambda} \quad (17)$$

where:

$$g(\mathbf{r}, t) \equiv \dot{\omega}_H - \dot{\omega} - \dot{k_z} v_z + k_z A_2,$$

$$\lambda \equiv \int k_x dx \quad (18)$$

and where

$$\alpha_1 = \omega_2/\omega_1, \alpha_2 = \frac{e}{m}\frac{E_z^w}{(\omega_1 v_\perp)}$$

and

$$\omega_{\tau 0}^2 = \omega_1 k_z v_\perp$$

The first term on the right-hand side of (17), $g(\mathbf{r}, t)$, depends upon the temporal and spatial variations of ω_H, ω, and k along the particle trajectory and in principle can be arbitrarily small. However, the second term on the right-hand side of (17) corresponds to the time derivative of (14) and can be large. To eliminate this large driving term from the right-hand side of (17) we make the substitution: $\gamma = \eta - \lambda$. In this case (17) becomes

$$\ddot{\eta} + \omega_{\tau 0}^2[\sin(\eta - \lambda) + \alpha_1 \sin(\eta + 2\Phi - \lambda) + \alpha_2 \sin\Phi] = g(\mathbf{r}, t) \quad (19)$$

Since all terms on the right-hand side of (19) now can be made arbitrarily small, (19) may admit solutions for which η satisfies a relation similar to (12) over the course of a few trapping periods. Clearly if η satisfies (12), $\underline{\eta}$ will also satisfy (12) (the bar under the symbol indicates an average over one gyroperiod). As we show below, $\underline{\lambda} = 0$. Thus $\underline{\gamma} = \underline{\eta}$ and $\underline{\gamma}$ must satisfy (12) whenever η does.

From (7) and (18) it can be shown that η obeys the equation:

$$\dot{\eta} = \omega_H - \omega - k_z v_z$$

which is formally the same relation obeyed by the phase angle γ in the case of propagation parallel to $\mathbf{B}_0$ [Dysthe, 1971; Bell and Inan, 1981]. As we show below, $\eta \simeq \underline{\eta} = \underline{\gamma}$. Thus physically the phase angle η can be conceived as the value of γ averaged over one gyroperiod.

To proceed further, we integrate (7) directly ($A_1 \simeq 10^{-4}\omega_H$ and is neglected) to obtain the relation:

$$\gamma = \omega_H t - \Phi + \gamma_0 + \Phi_0 \quad (20)$$

where γ_0 and Φ_0 are the initial values of γ and Φ at time $t = 0$. We can use (20) to find the form of λ:

$$\lambda \equiv \int k_x dx = \int k_x v_x dt = \beta \sin(\omega_H t + \theta_0) - \nu_0 \quad (21)$$

where $\theta_0 \equiv \gamma_0 + \Phi_0, \nu_0 \equiv (\beta \sin\theta_0 - k_x x_0)$, x_0 is the initial value of x for the particle, and where the last equality follows since we assume that k_x and $v_\perp$ do not vary significantly over a gyroperiod. Without loss of generality, the origin of our cartesian coordinate system can always be translated along the x axis to a point where $\nu_0 = 0$. In what follows we assume this translation has been made.

We now substitute (20) and (21) into (19) and use the well-known identity:

$$e^{i\beta \sin x} = \sum_{m=-\infty}^{\infty} J_m(\beta) e^{imx} \quad (22)$$

where $J_m(\beta)$ is the Bessel function of the first kind of order m and argument β.

These transformations result in the relation

$$\ddot{\eta} + \omega_\tau^2 \sin\eta = h(z) - \omega_{\tau 0}^2 F(\eta, t) - \alpha_3 \cos(2\sigma + 2\xi_0) \quad (23)$$

where $\alpha_3 \equiv (1/2) k_z \frac{v_\perp^2}{\omega_H}\frac{\partial \omega_H}{\partial z}$, $\sigma \equiv (\omega_H t + \theta_o)$ and where

$$F(\eta, t) = \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} [J_m(\beta) - \alpha_1 J_{m-2}(\beta) - \alpha_2 J_{m-1}(\beta)] \sin(\eta - m\sigma) \quad (24a)$$

$$h(z) = \dot{\omega}_H - \dot{\omega} - \dot{k}_z v_R + k_z \frac{v_\perp^2}{2\omega_H}\frac{\partial \omega_H}{\partial z} \quad (24b)$$

and

$$\omega_\tau^2 = \omega_{\tau 0}^2 [J_0(\beta) - \alpha_1 J_2(\beta) + \alpha_2 J_1(\beta)] \quad (24c)$$

In order to establish that phase trapping exists, it is sufficient to solve (23) for small variations of η. Expanding (23) in a two-term Taylor series in η about η_e, where $\eta_e = \sin^{-1}(h(z)/\omega_\tau^2)$, we obtain

$$\ddot{\eta} + [\omega_{\tau_e}^2 + q(t)](\eta - \eta_e) = -\omega_{\tau 0}^2 F(t) - \alpha_3 \cos(2\sigma + 2\xi_0) \quad (25)$$

where

$$q(t) = \omega_{\tau 0}^2 \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \ell_m(\beta) \cos(\eta_e - m\sigma)$$

$$F(t) = \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \ell_m(\beta) \sin(\eta_e - m\sigma) \quad (26)$$

$$\ell_m(\beta) \equiv [J_m(\beta) - \alpha_1 J_{m-2}(\beta) - \alpha_2 J_{m-1}(\beta)]$$

and

$$\omega_{\tau_e}^2 = \omega_\tau^2 \cos\eta_e = [\omega_\tau^4 - h^2(z)]^{1/2} \quad (27)$$

Equation (25) represents a linear differential equation with periodic coefficients of period $2\pi/\omega_H$. It can be shown that (25) admits bounded solutions for η so long as the condition holds:

$$m\omega_{\tau_e} \neq n\omega_H \quad (28)$$

where m and n are integers. If (28) is violated, a subharmonic resonance is possible in which $|\eta|$ will eventually exceed π. Since typically $\omega_{\tau_e} \simeq 10^{-2}\omega_H$, the subharmonic resonance will be of order 10^2 or higher and η will grow very slowly. Since ω_{τ_e} will vary in magnitude as the resonant particle moves along $\mathbf{B}_0$, (28) can be violated only over a few trapping periods at most. Thus subharmonic resonance will not be important. Assuming (28) holds, an approximate solution to (25) can be obtained through order $\epsilon = |B_w/B_0|$ by using the Krylov-Bogoliubov method [*Minorsky*, 1962]. This solution has the form

$$\eta = \eta_e + A\cos\Delta$$
$$-\omega_{\tau_e}^2 \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \left[\ell_m \frac{\sin(\eta_e - m\sigma)}{\omega_{\tau_e}^2 - m^2\omega_H^2} \right.$$
$$\left. +A\left(\frac{\ell_m + \ell_{-m}}{2}\right)\frac{\cos(\Delta + m\sigma - \eta_e)}{\omega_{\tau_e}^2 - (\omega_{\tau_e} + m\omega_H)^2}\right] \quad (29)$$

where $\Delta = \omega_{\tau_e}t + \delta$, $\sigma = \omega_H t + \theta_0$, and where A and δ are arbitrary constants. It can be seen that all terms within the summation sign are of order ϵ and it can be shown that the summation itself is of order ϵ. Thus to a good approximation the solution for η is given by the first two terms on the right-hand side in (29), and it is seen that for small oscillations η does not vary significantly during one gyroperiod.

The solution for η in (29) is stable and phase trapping is possible as long as ω_{τ_e} is real and positive. This condition can be expressed:

$$|\omega_\tau^2| > |h(z)| \quad (30)$$

The angular quadrant of the equilibrium phase angle η_e depends upon the sign of ω_τ^2. For $\omega_\tau^2 > 0$:

$$\eta_e^+ = \sin^{-1}\kappa \quad (31)$$

while for $\omega_\tau^2 < 0$:

$$\eta_e^- = \pi + \sin^{-1}\kappa \quad (32)$$

where $\kappa = h(z)/|\omega_\tau^2|$ and where the values of the arcsines are taken to range between $-\pi/2$ and $\pi/2$.

It can be seen from (31) and (32) that the angles η_e^+ and η_e^- will be 180° out of phase if the ratio κ is invariant. With the aid of (32) and (27) it can be shown that $\omega_{\tau_e}^2$ is never negative and that the second equality in (27) follows from the first.

2.2. Phase Trapping Equation

Equation (29) shows that for small oscillations the components of η that vary at frequencies of ω_H or higher can be neglected. It can be shown that this result is true for arbitrary values of η, thus we can average (23) over a gyroperiod to obtain the equation

$$\ddot{\eta} + \omega_\tau^2 \sin\eta = h(z) \quad (33)$$

where we have made use of the fact that $\underline{\eta} = \eta$ through order ϵ. Approximate solutions to (33) for values of $|\eta| < \pi$ can be obtained through the Krylov-Bogoliubov method [*Minorsky*, 1962] and it can be shown that the amplitude of the high-frequency components of η, $\omega = m\omega_\tau$ are of order $(m!)^{-1}$. Thus terms for which $m \approx \omega_H/\omega_\tau \approx 100$ will be negligible and η will not vary significantly during a gyroperiod. This finding is consistent with the assumption made in the averaging process that leads to (33).

Equation (33) is similar in form to that developed in earlier work concerning the case of phase trapping of energetic electrons by ducted whistler mode waves [e.g., *Dysthe*, 1971]. However, there are a number of important differences. First, the trapping frequency ω_τ in (33) can vanish for certain values of wave normal angle ψ and resonant particle pitch angle α. When $\omega_\tau = 0$, no phase trapping is possible. The condition $\omega_\tau \Rightarrow 0$ occurs whenever the force along $\mathbf{B}_0$ exerted by the wave on the particle averages out to zero. The vanishing of this average force is made possible because of the displacement of the particle transverse to $\mathbf{B}_0$ across planes of constant phase of the wave (see Figure 2a). This phenomenon is discussed further in section 4.

Whenever ω_τ^2 is positive, the conclusions of earlier papers dealing with ducted waves [e.g., *Dysthe*, 1971; *Inan et al.*, 1978; *Bell and Inan*, 1981] can be used directly, and it can be shown that η will be bounded and phase trapping for gyroresonant particles can exist in an average sense so long as (30) holds. In this case the phase of the most strongly trapped particles oscillates with period $2\pi/\omega_{\tau_e}$ about the equilibrium angle η_e^+, as given in (31). Furthermore it can be shown that resonant particles with $\eta \approx \pi + \eta_e^+$ are not phase trapped [Bell and Inan, 1981].

However, in the case in which ω_τ^2 is negative, the phase of the most strongly trapped particles oscillates about the equilibrium angle η_e^-, as given in (32), and it can be shown that resonant particles with $\eta \approx \pi + \eta_e^-$ are not phase trapped [Bell and Inan, 1981].

As a result of these considerations if we hold the magnitude of ω_τ^2 constant while changing its sign we find that for $\omega_\tau^2 < 0$ the most strongly trapped particles have phase angles that would be associated only with nontrapped particles in the case of interactions with ducted waves [Bell and Inan, 1981]. Furthermore, for $\omega_\tau^2 < 0$ the particles with $\eta \approx \eta_e^+$ are non trapped particles, whereas in the case of interactions with ducted waves these phase angles would be associated with the most strongly trapped particles. Because of these differences we henceforth refer to the phase trapping that takes place when $\omega_\tau^2 < 0$ as "anomalous" phase trapping (APT). For fixed values of ψ, the parameter β of the Bessel functions contained in the expression for ω_τ^2 can be made arbitrarily large by increasing the particle pitch angle α. Thus there is always a fraction of the resonant particles that will experience APT. As we show below, this effect can be important over a wide range of wave frequency, wave normal direction, and resonant particle pitch angle.

Another important difference in (33) is the fact that ω_τ depends upon E_z^w, while in the case of wave propagation along $\mathbf{B}_0$, ω_τ depends only upon B_w. As we will show below, the effects of E_z^w can dominate the phase trapping process for wave normals near the resonance cone.

2.3. Gradients in ψ and Wave Amplitude Threshold Values

A third major difference of (33) is the fact that the function $h(z)$ depends upon the gradient of the wave normal angle ψ through the term $\dot{k}_z v_R$, while in the case of ducted waves it is assumed that the gradient of ψ is always zero. To illustrate the implications of this difference we evaluate $h(z)$ explicitly by using the quasilongitudinal approximation to the whistler mode dispersion relation [*Helliwell*, 1965] for which

$$n(\psi) = [\frac{X}{Y_L - 1}]^{1/2} \quad (34a)$$

$$v_R = v_R(\psi) = c(Y-1)[\frac{Y_L - 1}{X}]^{1/2}[\cos\psi]^{-1} \quad (34b)$$

$$\begin{aligned} h(z) = h(z,\psi) = &\frac{\partial\omega_H}{\partial z}[1 + \frac{Y-1}{2(Y_L-1)}\cos\psi \\ &+ \frac{Y-1}{2Y}\tan^2\alpha]v_R - 1/2\frac{\partial\omega_0}{\partial z}(Y-1)X^{-1/2}v_R \\ &- (\frac{\partial\omega}{\partial t} + v_R\frac{\partial\omega}{\partial z})\left[1 + \frac{Y_L(Y-1)}{2(Y_L-1)}\right] \\ &+ (\frac{\partial\psi}{\partial t} + v_R\frac{\partial\psi}{\partial z})\left[\frac{(Y_L-2)(Y-1)}{2(Y_L-1)}\right]\omega\tan\psi \end{aligned} \quad (34c)$$

where $Y = \omega_H/\omega$, $Y_L = Y\cos\psi$, $X = (\omega_0/\omega)^2$, and where we have assumed that ω_H and ω_0 have no explicit time dependence.

With the aid of (34c) let us examine how gradients in ψ can affect the threshold value of B_w necessary for phase trapping near the magnetic equatorial plane. For the commonly used dipole model for ω_H and diffusive eqiulibrium (DE) model for ω_0 [*Angerami and Thomas*, 1964], $\frac{\partial\omega_H}{\partial z} \Rightarrow 0$ and $\frac{\partial\omega_0}{\partial z} \Rightarrow 0$ at the magnetic equatorial plane and thus for ducted waves of constant frequency at the equator, $h(z = 0) = 0$. The vanishing of $h(0)$ does not imply however that waves of vanishingly small amplitude can satisfy (30), since (30) must hold over a time comparable to the trapping period and during this time the resonant electron will move from the equator to regions where gradients in ω_H and ω_0 do not vanish. Thus there is a threshold value of B_w, B_w^{min}, near the magnetic equator below which phase trapping will not occur. Since $\partial\omega_H/\partial z >> \partial\omega_0/\partial z$ for the DE model, B_w^{min} is determined principally by the variation of ω_H. In this case, as shown in the work of *Inan et al.* [1978], near the equatorial plane B_w^{min} satisfies (30) with $h(z) = h(z_2)$ where

$$z_2 = [\frac{4v_R r^2}{3[3\omega_{Ho} + (\omega_{Ho} - \omega)\tan^2\alpha]}]^{1/3} \quad (35)$$

and where $\omega_{Ho} = \omega_H(z = 0)$ and r is the geocentric radius to the position of the resonant particle.

Using (35) and (30) we find

$$B_w^{min}(\psi = 0) = 7.6 \times 10^3 \times \cot\alpha\, Y(Y-1)^{1/3} \times \frac{[3/2 + \frac{(Y-1)}{Y}\tan^2\alpha]^{2/3}}{L^{4/3}\omega_{Ho}^{1/3}n^{4/3}}\; milligamma \quad (36)$$

where ω_{Ho} is measured in rad/sec and $n = n(\psi = 0)$ is given in (34a).

For $L \approx 4$, $\omega_{Ho} \sim 9 \times 10^4$ rad/sec, $\omega_0 \sim 10^6$ rad/sec and

$$B_w^{min} = 0.4\cot\alpha Y^{-1/3}(Y-1)[3/2 + \frac{Y-1}{Y}\tan^2\alpha]^{2/3}\; milligamma \quad (37)$$

In general, the threshold values given by (37) are modest. For example, for moderate pitch angles,e.g., $\alpha \approx 45°$, (37) yields $B_w^{min}(0) = 0.5m\gamma$ for $Y = 2$ and $B_w^{min}(0) = 3m\gamma$ for $Y = 10$.

In the case of nonducted waves of fixed frequency near the magnetic equator, we have from (34c)

$$h(0,\psi) \approx \frac{\partial\psi}{\partial z}[\frac{(Y_L-2)(Y-1)}{2(Y_L-1)}]v_R\omega\tan\psi \quad (38)$$

It is not possible to use (38) directly to calculate a threshold value of B_w for nonducted waves since at the present time no adequate model exists for the variation of ψ along a resonant particle trajectory.

However, *Edgar* [1976] has given a ray tracing equation that governs the behavior of ψ along a ray trajectory. If we assume that the DE model describes the variation of the cold plasma density N along $\mathbf{B_0}$ and that the variation of N across $\mathbf{B_0}$, has the form $N \sim r^{-s}$, where r is the radius from the earth's center, then the equation for ψ near the magnetic equatorial plane has the form:

$$\frac{2n(\psi)}{c}\dot{\psi} = \frac{\cos\psi}{r}[s - 3(2 + \frac{Y_L}{Y_L - 1})] \quad (39)$$

where $n(\psi)$ is given in (34a).

Since by assumption $\dot{\omega} = 0$,then $\dot{\psi} = \mathbf{v}_g \cdot \nabla\psi$ where $\mathbf{v}_g$ is the group velocity and ∇ is the spatial divergence operator. For $Y \geq 2$, the angle between $\mathbf{v}_g$ and $\mathbf{B_0}$ is always less than 30° [*Helliwell*, 1965]; thus $d\psi/dt \approx v_g\partial\psi/\partial z$.

Since $v_g \approx 2c/n(\psi)(Y_L - 1)/Y_L$ for whistler-mode waves [*Helliwell*, 1965], we can rewrite (39):

$$\frac{\partial\psi}{\partial z} \approx \frac{Y_L\cos\psi}{4(Y_L-1)r}[s - 3(2 + \frac{Y_L}{Y_L-1})] \quad (40)$$

Equation (40) must be used with caution since it describes the variation of ψ along the ray path and not along $\mathbf{B_0}$. In general, each point along $\mathbf{B_0}$ will be intersected by a different ray and the variation of ψ between the ray intersections gives the true value of $\frac{\partial\psi}{\partial z}$ experienced by the resonant particle. However, if the optical path length of adjacent rays is approximately the same, then (40) should be a reasonable approximation to the variation of ψ along $\mathbf{B_0}$. It is clear from (40) that the variation of ψ along $\mathbf{B_0}$ can be zero only when the term within the brackets is zero. This can only occur for very sharp radial decreases in N, e.g., $N \sim r^{-9}$ for $Y >> 1$, and $N \sim r^{-20}$ for $Y = 2$ and $\psi = 45°$.

Such rapid dropoffs could be found locally near the plasmapause or near $L \sim 2$ [*Edgar*, 1976], but the general radial decrease in N within the plasmasphere is much slower [*Park et al.*,1978]. Thus in general the radial gradient of N can be neglected in (40).

Equations (30), (38), and (40) can be used to estimate the minimum value of B_w necessary for phase trapping to take place near the magnetic equator:

$$B_w^{min} \simeq \frac{400\sin\psi}{L(1+\cos\psi)}(\frac{\omega_H}{\omega_0})(2 + \frac{Y_L}{Y_L-1}) \frac{(Y-1)(Y_L-2)\cot\alpha}{(Y_L-1)^{3/2}[J_0(\beta) - \alpha_1 J_2(\beta) + \alpha_2 J_1(\beta)]}\; milligamma \quad (41)$$

Equation (41) is plotted in Figure 9 for plasma parameters appropriate to $L \approx 4$, i.e., $N \approx 400cm^{-3}$ and $\omega_H/\omega_0 \approx 0.08$, assuming $\alpha \approx 45°$. Two values of Y are considered, $Y = 10$ and $Y = 2$. These plots will be discussed in section 3.

2.4. Time-Averaged Equations

Equation (29) shows that η does not vary significantly during any one gyroperiod. Thus, making the substitution

$\gamma = \eta - \lambda$ in (7)–(9) and averaging over one gyroperiod an approximate set of time-averaged equations can be derived for the particle motion:

$$\dot{\eta} = \omega_H - \omega - k_z v_z \tag{42a}$$

$$\dot{v}_z = k_z^{-1}\omega_\tau^2 \sin\eta - \frac{v_\perp^2}{2\omega_H}\frac{\partial \omega_H}{\partial z} \tag{42b}$$

$$\dot{v}_\perp = -[\omega_1(v_z + R_1)J_0(\beta) - \omega_2(v_z - R_2)J_2(\beta)]\sin\eta + \frac{v_\perp v_z}{2\omega_H}\frac{\partial\omega_H}{\partial z} \tag{42c}$$

where now the dependent variables v_z, $v_\perp$, and η are understood to be time-averaged quantities. In deriving (42), we have made use of the fact that $\eta = \underline{\eta}$ through order ϵ and thus $\underline{\sin\eta} = \sin\eta$ to this order.

The equations in (42) are useful in determining the average trajectories of energetic electrons moving under the influence of the nonducted wave and can be used to reduce computation time in computer simulation studies of interactions between nonducted waves and energetic electrons in the magnetosphere. Simulation studies using both (7)–(9) and (42) need to be carried out to assess the accuracy of (42).

2.5. Energy and Pitch Angle Changes

In the above we have shown that cyclotron resonant electrons can be phase trapped in an average sense by nonducted whistler mode waves. This finding will be relevant to magnetospheric wave-particle interactions only insofar as significant cumulative changes in particle energy and/or pitch angle are associated with this average phase trapping.

Furthermore, in application to the mechanism of VLF emission generation it is important to determine if all strongly phase trapped (on the average) particles radiate in phase.

When the perturbing wave propagates parallel to $\mathbf{B}_0$, $\lambda \Rightarrow 0$, $\gamma \Rightarrow \eta$, and it can be shown from (33) and (31) that all strongly phase-trapped electrons ($\eta \approx \eta_e$) will constitute a current perpendicular to $\mathbf{B}_0$ that will radiate in phase with the perturbing wave [*Helliwell*, 1967]. However, for the general case of $\psi \neq 0$, $\gamma \neq \eta$, and is not clear that particles which are strongly phase-trapped in an average sense and obey (33) will all radiate in phase with the wave. This question becomes even more pressing in view of (32), which indicates that the equilibrium position of strongly phase-trapped particles will shift $\sim 180°$ as ω_τ^2 changes sign.

In general, the kinetic energy change (per unit mass) of each resonant electron will obey the relation:

$$\dot{K} = \frac{d}{dt}(1/2v^2) = -\frac{e}{m}[\mathbf{v}\cdot\mathbf{E}_w] \tag{43}$$

where $\mathbf{E}_w$ is given in (2).

Using (20) and the definitions of v_x and v_y (see Figure 2), and letting $v_z = v_R$ (as given in (13)), an average value of $\dot{K}$ over one gyroperiod can be obtained:

$$\underline{\dot{K}} = -\frac{e}{m}[\frac{v_\perp}{2}\{E_x^w[J_0(\beta) + J_2(\beta)] + E_y^w[J_0(\beta) - J_2(\beta)]\} - E_z^w v_R J_1(\beta)]\sin\eta \tag{44}$$

From Maxwell's equations it can be shown that

$$B_x^w = B_y^w \cos\psi = \frac{k_z}{\omega}E_y^w$$

$$B_y^w = \frac{k_z}{\omega}E_x^w - \frac{k_x}{\omega}E_z^w \tag{45}$$

With the aid of (45), (44) can be rewritten:

$$\underline{\dot{K}} = -\frac{\omega}{k_z^2}\omega_\tau^2 \sin\eta \tag{46}$$

where ω_τ^2 is given by (24c) and where in deriving (46) we have made use of the recursion relation

$$J_1(\beta) = \frac{(\beta)}{2}[J_0(\beta) + J_2(\beta)].$$

From (46),(31), and (32), it is clear that the sign of $\underline{\dot{K}}$ is the same for all particles whose average phase is near one of the equilibrium points, $\eta \approx \eta_e^+$, or $\eta \approx \eta_e^-$. Thus all strongly phase-trapped particles will tend to radiate in phase, even those subject to APT.

The time-averaged pitch angle change of resonant particles can be obtained from either (7)–(9) or (42), and the expression has the form:

$$\dot{\alpha} = -\frac{\omega_\tau^2(1 + \frac{\cos^2\alpha}{Y-1})}{k_z v_\perp}\sin\eta + \frac{v_\perp}{2\omega_H}\frac{\partial\omega_H}{\partial z} \tag{47}$$

It can be seen from (47) that when $\omega_\tau \Rightarrow 0$ (i.e., when the average force along $\mathbf{B}_0$ exerted by the wave vanishes) the pitch angle change of the particle is due only to the adiabatic mirror force. However, since $\omega_\tau \Rightarrow 0$ only for specific values of α, it is clear that the adiabatic mirror force can transport the resonant particle either into, or out of, regions where $\omega_\tau \approx 0$.

2.6. Poynting Flux

In the calculations to follow, we need to relate B_y^w to the total energy carried by the perturbing whistler mode wave. From the usual definition of the Poynting flux we find:

$$S(\psi) = |\mathbf{E}_w \times \mathbf{H}_w| = \frac{c}{2\mu_0}\frac{|B_y^w|^2\cos^2\psi}{\eta(\psi)} \times \{[1 + \rho_2^2 - \rho_1\rho_2\tan\psi]^2 + [tan\psi(1+\rho_1^2) - \rho_1^2]^2\}^{1/2} \tag{48}$$

where $\eta(\psi)$ is given in (34a) and $\rho_{1,2}$ are given by the ratios:

$$\rho_1 \equiv \frac{E_z^w}{E_y^w} = \frac{sin\psi}{(Y_L - 1)}$$

$$\rho_2 = \frac{E_x^w}{E_y^w} = \frac{Y - cos\psi}{(Y_L - 1)}$$

where the explicit values have been derived within the context of the quasi-longitudinal approximation as given in (34a).

2.7. General Harmonic Resonance

It is of interest to determine the extent of the phase bunching that exists for other harmonics of ω_H. For a har-

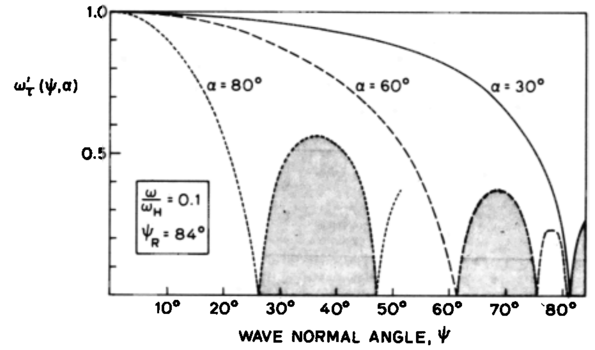

Fig. 3. Normalized phase trapping frequency ω_T' plotted as a function of wave normal angle ψ for resonant particles of three different pitch angles, $\alpha = 30°, 60°, 80°$. A normalized frequency of 0.1 is assumed.

monic of order m we have the general resonance condition:

$$v_{Rm} = \frac{m\omega_H - \omega}{k_z} \quad (49)$$

Making the substitution $\gamma_m = \eta_m - (m-1)(\omega_H t + \theta_0) - \lambda$ in (17), using (20), (21), and (22), and averaging the resultant equation over one gyroperiod, we obtain:

$$\ddot{\eta}_m + \omega_T^2 \sin \eta_m = h_m \quad (50)$$

where

$$\omega_{Tm}^2 = (-1)^{m-1}\omega_{T0}^2[J_{m-1}(\beta) - \alpha_1 J_{m+1}(\beta) + \alpha_2 J_m(\beta)] \quad (51)$$

and where h_m is equal to the right-hand side of (34c) with each factor $(Y-1)$ replaced by $(mY-1)$ and each factor v_R replaced by v_{Rm}, where v_{Rm} is given in (49), and where

$$\beta \equiv (mY-1)Y^{-1} \tan\alpha \tan\psi.$$

When $m = 0$, (50) describes the Landau resonance interaction. The existence of a phase trapping equation for the Landau resonance might appear surprising at first sight, but it is clear through (20) that $\eta_{m=0}$ can be considered as a normalized measure of the position of the resonant particle along the z axis. In fact, using (20) it can be shown that for $m = 0$ (50) reduces to the spatial bunching equation appropriate to the Landau resonance [*Inan and Tkalcevic*, 1982].

3. CALCULATIONS

3.1. The Trapping Frequency

When $h(z) \neq 0$, it is the magnitude of the trapping frequency ω_T that determines if phase trapping is possible, in accordance with (30). Thus it is of interest to determine ω_T as a function of wave frequency ω, wave normal angle ψ, and resonant particle pitch angle α.

Figure 3 illustrates the variation of normalized trapping frequency ω_T' (normalized to its value at $\psi = 0$) with wave normal angle for a wave frequency of 1/10 the local electron gyrofrequency. Curves corresponding to three values of resonant particle pitch angle α are shown. In calculating values of ω_T' for all curves the wave power flux was held constant. Thus the value of B_y^w used was obtained from (48)

$$B_y^w(\psi) = B_y^w(\psi = 0)[\frac{S(\psi = 0)}{S(\psi)}]^{1/2} \quad (52)$$

For a pitch angle of 30°, the trapping frequency does not vary significantly from its value at $\psi = 0$, until $\psi = 60°$. The first zero of $\omega_T'(\alpha = 30°)$ occurs at $\psi \simeq 82°$, and the first region of anomalous phase trapping (APT) occurs between 82° and the resonance cone at $\psi \sim 84°$. The portions of the curve corresponding to APT are indicated by crosshatching between the curve sections and the ψ axis.

For $\alpha = 60°$, $\omega_T' \Rightarrow 0$ at a number of points along the ψ axis and three of these intersections are shown in the plot. The first region of APT occurs between the first two zeros of ω_T' at $\approx 61°$ and $\approx 76°$, respectively. The region between the second zero and the third zero is one of normal phase trapping. This illustrates the general rule that regions of normal and anomalous phase trapping are always separated by zeros of ω_T' and always occur alternately along the ψ axis. For simplicity, only the first three regions of the $\alpha = 60°$ plot are shown.

For $\alpha = 80°$, $\omega_T' \Rightarrow 0$ at a number of points along the ψ axis, but for simplicity only two are shown. The first zero occurs at $\psi \approx 26°$ and the first region of APT extends from $\psi \approx 26°$ to $\psi \approx 47°$.

Clearly, for low normalized frequencies and for moderate to high particle pitch angles. regions of APT can exist over a wide range of wave normal angles.

Figure 4 illustrates the variation of ω_T' with ψ for a case in which wave frequency is equal to one-half of the local electron gyrofrequency. For $\alpha = 30°$, the trapping frequency is relatively independent of ψ for $\psi < 50°$. However, as the wave normal angle approaches the resonance cone angle ($\psi_R = 60°$), there is a rapid increase in ω_T'. This increase is due to the effects of E_z^w, the component of the wave electric field parallel to B_o. As $\psi \Rightarrow \psi_R$, there is relatively more energy stored in the electric field of the wave than in the magnetic field. Since we have fixed the wave power, this redistribution of energy allows E_z to dominate the interaction and increase the trapping frequency as $\psi \Rightarrow \psi_R$. There are no regions of APT for $\alpha = 30°$, nor for any $\alpha < 75°$.

For $\alpha = 80°$, a single region of APT exists for the range $45 \leq \psi \leq 60°$. The increase in ω_T' near the resonance cone angle, $\psi_R = 60°$, is again due to the effects of E_z^w. For $\alpha =$

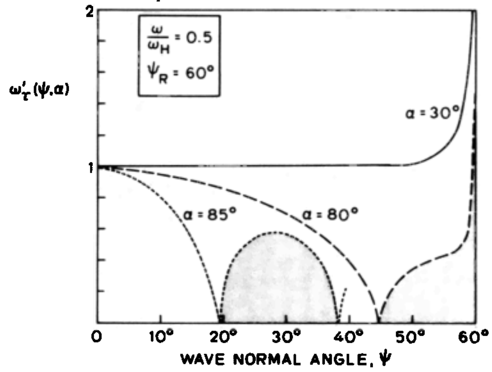

Fig. 4. Normalized phase trapping frequency ω_T' plotted as a function of wave normal angle ψ for resonant particles of three different pitch angles, $\alpha = 30°, 80°, 85°$. A normalized frequency of 0.5 is assumed.

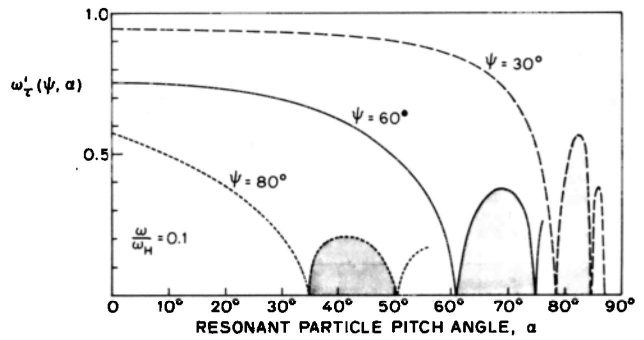

Fig. 5. Normalized phase trapping frequency ω'_T plotted as a function of resonant particle pitch angle α for three values of wave normal angle, $\psi = 30°, 60°, 80°$. A normalized frequency of 0.1 is assumed.

85°, a number of regions of APT exist, but for simplicity only the first is shown, extending from $\psi \approx 20°$ to $\psi \approx 38°$.

Figures 5 and 6 show the variation of ω'_T as a function of the resonant particle pitch angle for various fixed values of wave normal angle. These plots show details which could not be illustrated in Figures 3 and 4 because of their complexity. Figure 5 represents the case of low normalized wave frequency ($\omega = 0.1\omega_H$) and gives results for three values of wave normal angle. As in previous plots, regions of APT are indicated by cross-hatching under the curve. In this representation each curve of ω'_T has an infinite number of zeros and an infinite number of regions of normal phase trapping and anomalous phase trapping, which alternate along the axis. For simplicity, only the first few regions are shown. Since ω'_T is normalized to the value of ω_T at $\psi = 0$, the curve does not approach unity as $\alpha \Rightarrow 0$.

It is clear that the regions of APT can involve a significant number of resonant particles for $\psi \geq 60°$.

Figure 6 represents the case of moderate normalized wave frequency ($\omega = 0.5\omega_H$). For low wave normal angles ($\psi \leq 30°$) ω'_T varies very little with α for $\alpha < 75°$ and the first zero does not occur until $\alpha \approx 84°$. For $\psi = 57°$, the wave normal is within 3° of the resonance cone angle but, nevertheless, ω'_T varies very little for $\alpha < 60°$, and the first zero is not reached until $\alpha \approx 77°$. For $\psi = 57°$, the trapping frequency exceeds unity for $\alpha < 75°$. Thus the trapping frequency has increased with ψ, in contrast to the situation in Figure 5, where ω'_T decreases with ψ. The increase in ω'_T with ψ is caused by the wave electric field component E^w_z. The dotted curve in Figure 6 shows the value of the trapping frequency for $\psi = 57°$ when E^w_z is arbitrarily set equal to zero in (24c). Comparison of the two curves for $\psi = 57°$ shows that the effect of E^w_z is to increase ω'_T and to translate the first zero from ω'_T from $\alpha \approx 75°$ to $\alpha \approx 77°$. Calculations show that a similar situation holds for all wave frequencies in the range $\omega_H/2 \leq \omega \leq \omega_H$. Thus for moderate to high normalized wave frequency, the main effect of the wave electric field component E^w_z is to enhance the strength of the phase trapping mechanism inside the first region of normal phase trapping.

3.2. The Strength of Phase Trapping

The magnitude of the trapping frequency (for given values of α and ψ) does not in itself determine if local phase trapping can exist since (30) must hold. Thus it is more useful to speak in terms of the parameter $\kappa \equiv h(z)/|\omega_T^2|$ when discussing the relative strength of phase trapping interactions at different values of ψ and α. Such a parameter (or a similar parameter) has been used by earlier workers to categorize phase trapping that takes place with ducted waves [*Dysthe*, 1971; *Nunn*, 1974; *Karpman et al.*, 1974a,b; *Roux and Pellat*, 1978; *Bell and Inan*, 1981]. In comparing phase trapping interactions occurring in ducted and nonducted waves, it is useful to determine the relative wave power in the two cases which is necessary to achieve the same value of κ. To perform the calculations, the ratio κ was computed from (24c) and (34c) and was normalized to its value at $\psi = 0$. The wave magnetic field ratio $B^w_y(\psi)/B^w_y(0)$, necessary to achieve the relation $\kappa' = \kappa(\psi)/\kappa(0) = 1$, was determined for each ψ and this ratio was used in (48) to obtain the relative power.

It was assumed that the cold plasma density in the magnetosphere was described by the well-known diffusive equilibrium model of *Angerami and Thomas* [1964]. For this model $\partial\omega_o/\partial z << \partial\omega_H/\partial z$. In addition, the term in $h(z)$ proportional to $\partial\psi/\partial z$ was neglected because of the lack of an adequate model for the quantity. Neglect of this term may be reasonable at high magnetic latitude where gradients in ω_H become large. However, we shall consider this term in

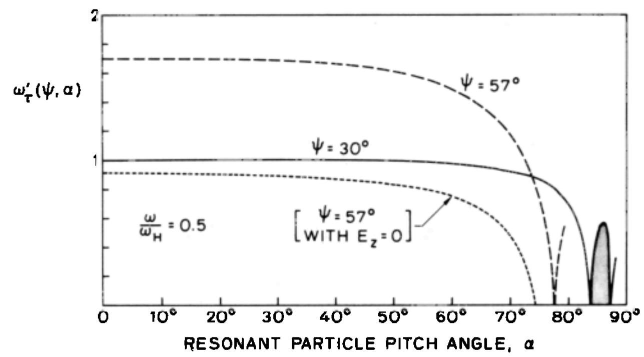

Fig. 6. Normalized phase trapping frequency ω'_T plotted as a function of resonant particle pitch angle α for two values of wave normal angle, $\psi = 30°, 57°$. A normalized wave frequency of 0.5 is assumed.

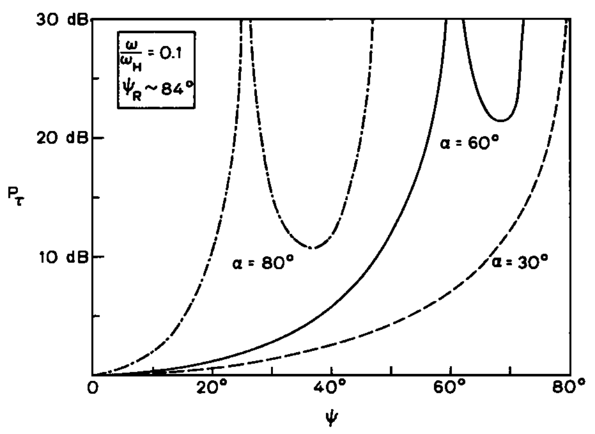

Fig. 7. Relative wave power P_T necessary to maintain constant phase trappping efficiency as a function of wave normal angle ψ for three values of resonant particle pitch angle $\alpha = 30°, 60°, 80°$. A normalized wave frequency of 0.1 is assumed, and it is also assumed that gradients in ψ are negligible.

more detail below. The remaining terms in $h(z)$ are proportional to $\partial\omega_H/\partial z$, a function which does not depend upon ψ. Thus the ratio κ' was independent of the local gradient of ambient magnetic field.

Figure 7 shows the relative wave power necessary at each value of ψ to achieve a fixed value of κ', in the case of low normalized wave frequency. These plots are typical of all those for which $\omega/\omega_H \leq 0.1$. For each curve, calculations are shown only up to the second zero of ω_τ.

For moderate wave normal angles ($\psi \approx 40°$), particles with $\alpha \leq 60°$ can be phase trapped with a wave power increase of only 5 dB. However, for high wave normal angles ($\psi > 60°$) and/or high pitch angles, power increases of up to 30 dB or more are necessary.

This situation is much different for the case of moderate to high wave frequency. The plots of Figure 8 are typical of those for which $0.5 \leq \omega/\omega_H < 1$. In general, little additional wave power is required to phase trap particles over a wide range of pitch angles. Since all points for $\alpha < 60°$ lie below the $\alpha = 60°$ curve, it is clear that only at rather high pitch angle does the required wave power increase exceed a few dB.

Figures 7 and 8 do not include the effects of the gradient of ψ along $\mathbf{B}_o$ since no model exists for this quantity. However, these effects can be estimated near the magnetic equator using (41). In Figure 9 we plot (41) for plasma parameters appropriate to $L \approx 4$, i.e., $N \approx 400\,cm^{-3}$ and $\omega_H/\omega_o \approx 0.08$, assuming $\alpha \approx 45°$. Two values of Y are considered, $Y = 10$ and $Y = 2$. For $Y = 10$, the threshold value of B_w necessary to achieve phase trapping exceeds $10m\gamma$ for $\psi > 17°$ and approaches $50m\gamma$ for $\psi \Rightarrow 50°$. For $Y = 2$, $B_w^{min} \leq 1m\gamma$ for $\psi \leq 20°$, but $B_w^{min} \Rightarrow 50m\gamma$ as $\psi \Rightarrow 45°$. Comparing the plots in Figure 9 with the threshold values given by (37) (the values are marked by the letter "T" in the figure) it is clear that near the magnetic equator gradients in ψ will be the most important factor in determining B_w^{min} so long as $\psi > 10°$ during the interactions. At higher latitudes, gradients in ω_H become larger and possibly dominate.

3.3. The Effects of E_z^w

It is clear from the form of (24c) that the wave parallel electric field will periodically dominate the phase trapping process in regions of parameter (ω, ψ, α) space in which the average magnetic forces approach zero. It is also clear that E_z^w will dominate the trapping process as $\psi \Rightarrow \psi_R$, where ψ_R is the resonance cone angle.

It is of interest to determine the importance of E_z in the first region of normal phase trapping where ω_τ has not yet gone to zero. One measure of the importance of E_z^w in this region is the ratio of the trapping force exerted by E_z^w to the maximum trapping force exerted by the wave magnetic field at $\psi = 0$. For fixed α, this ratio has the value:

$$\epsilon' = \frac{\sin\psi\cos\psi}{(Y-1)(Y_L-1)\tan\alpha} J_1(\beta) \tag{53}$$

The ratio ϵ' represents the extent to which the wave electric forces act to raise $\omega_\tau(\psi)$ toward its original value at $\psi = 0$. For any given ψ and ω, ϵ' has a maximum value for small values of α:

$$\epsilon'_m = \frac{\sin^2\psi}{2Y(Y_L-1)} \tag{54}$$

The curve corresponding to $\epsilon'_m >> 1$ is essentially a plot of the resonance cone angle as a function of ω. In Figure 10 we use (54) to plot the values of ω and ψ appropriate to three

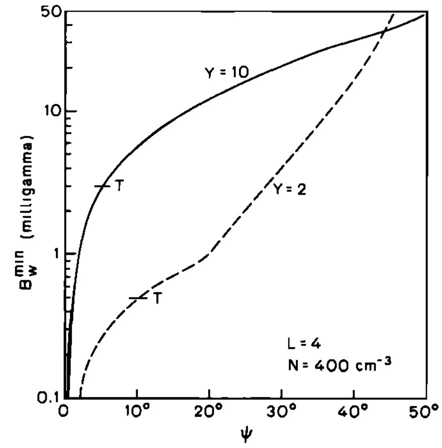

Fig. 9. Phase trapping wave amplitude threshold value plotted as a function of wave normal angle for two values of normalized frequency, 0.1 (Y = 10) and 0.5 (Y = 2). Interaction is assumed to take place close to the magnetic equatorial plane where only gradients in ψ are important.

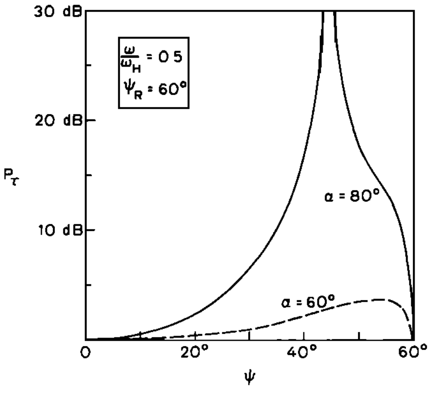

Fig. 8. Relative wave power P_τ necessary to maintain constant phase trapping efficiency as a function of wave normal angle ψ for two values of resonant particle pitch angle $\alpha = 60°, 80°$. A normalized wave frequency of 0.5 is assumed and it is also assumed that gradients in ψ are negligible.

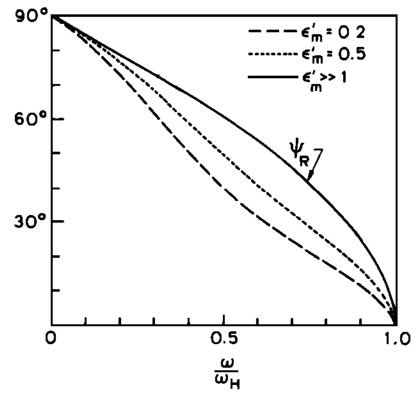

Fig. 10. Regions of $\psi, \omega/\omega_H$ space in which the trapping force due to the wave electric field parallel to B_o exceeds a given fraction ϵ of the maximum trapping force due to the wave magnetic field. Contours for $\epsilon = 0.2$, 0.5 and $>> 1$ are shown.

given values of ϵ'_m. It is clear from the plot that effects due to E_z^w can be important for values of ψ well removed from the resonance cone as long as $0.3 < \omega/\omega_H < 1$.

4. Discussion and Conclusions

4.1. Phase Trapping

The theory developed above predicts that a phase trapping mechanism can affect energetic electrons which are in cyclotron resonance with a coherent VLF wave propagating at an arbitrary angle ψ with respect to the earth's magnetic field. For small values of ψ and/or $\omega \approx \omega_H/2$, the phase trapping is similar to that of the case of ducted waves where $\psi = 0$. However, for large values of ψ and/or low values of ω, the phase trapping conditions can differ markedly from those of the case of ducted propagation. In particular, we have shown how the trapping frequency vanishes at a countably infinite number of values of particle pitch angle.

From (42b) and the well-known relation $\mathbf{F} = m\dot{\mathbf{v}}$ it is evident that ω_τ^2 is proportional to the average value (one gyroperiod) of the force along $\mathbf{B}_0$ exerted by the wave on the resonant particle. Thus the condition $\omega_\tau \Rightarrow 0$ occurs whenever the wave force along $\mathbf{B}_0$ averages out to zero. In order to see how this comes about it is instructive to consider in detail the temporal variation of the force on the particle due to the wave component $\mathbf{B}_R$. In Figure 11 we show the relationship between the vectors $\mathbf{v}_\perp$ and $\mathbf{B}_R$ for the case in which $\beta \simeq 80°$ and $\gamma_1 \simeq 30°$, where γ_1 is the value of the phase angle γ at the particle's initial position in space. The circle in Figure 11 is the approximate locus of the points representing the particles position in the $x - y$ plane during one gyroperiod, and at every instant the particle's perpendicular velocity vector $\mathbf{v}_\perp$ is tangent to this circle at the position of the particle. Without loss of generality we can choose the center of the circle to be located on the y axis. We assume that the particle is in gyroresonance with the wave so that (13), (14), and (15) hold.

At position 1 around the circle $\gamma = \gamma_1 \approx 30°$ and the

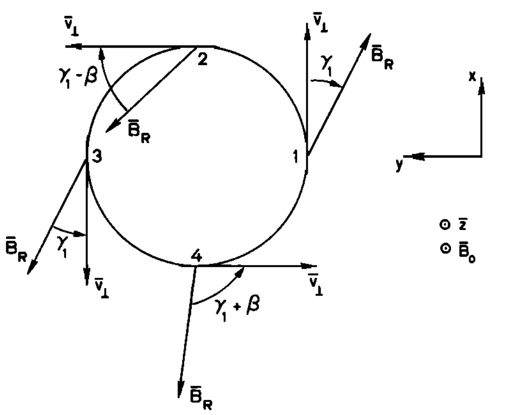

Fig. 11. Relationship between $\mathbf{v}_\perp$ and $\mathbf{B}_R$ during one gyroperiod for case in which $\gamma_1 \approx 30°$ and $\beta \approx 80°$, where γ_1 is the initial phase angle between $\mathbf{v}_\perp$ and $\mathbf{B}_R$. The quantity β represents the maximum wave phase advance (or retardation) due to the particle motion in the x direction across planes of constant phase. The circle is the locus of the points which represent the particles position in the $x - y$ plane. In this particular case the force along $\mathbf{B}_0$ exerted by $\mathbf{B}_R$ upon the resonant electron is positive at positions 1 and 3 and negative at positions 3 and 4, and the average force is approximately 1/2 that which would be obtained for a ducted wave ($\psi = 0$ and $\beta = 0$).

force $F_z = -e(\mathbf{v}_\perp \times \mathbf{B}_R)$, along $\mathbf{B}_0$ due to $\mathbf{B}_R$, is positive. At position 2 the particle has translated along the x direction to a position where the wave phase is advanced by the factor β, and $\gamma = (\gamma_1 - \beta) \approx -50°$. Here F_z is negative. At position 3, F_z is positive again and has the same value as that of position 1. At position 4, the particle has translated along the negative x direction to a position where the wave phase is retarded by the factor β, and $\gamma = (\gamma_1 + \beta) \approx 110°$. Here F_z is negative again. Averaging over one gyroperiod we find $\underline{F_z} = ev_\perp|\mathbf{B}_R|J_0(\beta)\sin\gamma_1$, where in this particular case $\beta \approx 80°$ and $J_0(80°) \approx 0.5$. Clearly for any value γ_1, $\underline{F_z}$ can vanish or change sign for certain values of β. Forces along $\mathbf{B}_0$ due to the wave components $\mathbf{B}_L$ and E_z have similar behavior, and the total average force exerted by the wave along $\mathbf{B}_0$ can vanish or change sign as β is varied.

In the vicinity of points at which $\omega_\tau(\alpha) \Rightarrow 0$, condition (30) will not be satisfied and the resonant particles will not be trapped. Using arguments similar to those developed in a previous paper [Bell and Inan, 1981], it can be shown that generally these resonant but nontrapped particles constitute a perpendicular current $\sim 180°$ out of phase with the perpendicular current due to the phase-trapped particles. Thus the currents due to the two sets of particles will tend to cancel, the radiation due to the total current will be diminished, and the feedback process between the wave and the resonant particles will be less efficient than in the case of ducted propagation [Helliwell and Crystal, 1973; Helliwell and Inan, 1982]. The feedback process will be even less efficient if the energetic particles have a pitch angle distribution which is sharply peaked toward 90° pitch angle, i.e., $f(\alpha) \sim \sin^m \alpha$. In the linear theory [Kennel and Petschek, 1965; Liemohn, 1967] such distributions are necessary to produce wave growth due to linear gyroresonance with energetic particles, and distributions with $m \approx 12$ have been observed in connection with naturally produced VLF emissions [Anderson and Maeda, 1977; Kimura et al., 1983]. However, if m is large, most gyroresonant electrons will possess high pitch angles and for moderate values of ψ will be located deep within the region of alternating normal and anomalous phase trapping zones. In this case a higher degree of cancellation between the currents from the trapped and nontrapped resonant electrons can be expected.

It is noteworthy that in the mid-frequency range ($\omega/\omega_H \approx 0.5$), the trapping frequency stays relatively constant up to high pitch angles for a wide range of wave normal angle (see Figure 6), and regions of anomalous phase trapping exist only at high pitch angle. This behavior indicates that at mid-frequency the phase trapping interaction for arbitrary ψ is similar to the phase trapping that occurs at $\psi = 0$. Thus, concepts developed in earlier work [Bell and Inan, 1981] for the $\psi = 0$ and $\omega/\omega_H \simeq 0.5$ case can apparently be carried over almost directly to the case of arbitrary ψ. However, one major difference between the case of parallel propagation and the case of arbitrary ψ is the fact that the parallel component of the wave electric field E_z^w can play an important part in the phase trapping process (see Figure 6).

The reason that E_z^w enters into the phase trapping process at all hinges upon the fact that, owing to its motion transverse to $\mathbf{B}_0$ across planes of constant wave phase, a gyroresonant particle experiences the parallel electric field as a frequency-modulated wave of instantaneous frequency $\omega' = \omega_H - \omega_H\beta\cos\omega_H t$.

This frequency variation corresponds to a frequency modulation in which the modulating frequency is equal to the carrier frequency. In this case the wave electric field experienced by the particle can be expressed as the sum of a "carrier" wave at frequency ω_H plus sideband waves located at frequencies $\omega = \omega_H + N\omega_H$ where $|N| = 1, 2, ..., $etc. The sideband wave at zero frequency ($N = -1$) has the amplitude $J_1(\beta)$ and is the only wave component for which the time average is not zero in general. Since for small β, $J_1(\beta) \approx \beta/2 = k_\perp v_\perp/2\omega_H$, the magnitude of phase trapping effects due to E_z^w will depend upon $v_\perp$ just as do effects due to $B_{w_\perp}$.

4.2. Gradients in ψ

At the present time no adequate model exists for the variation of ψ along a resonant particle trajectory (along $\mathbf{B}_0$); consequently, it is not possible to establish the relative contribution of gradients in ψ in setting the amplitude threshold value for phase trapping. However, since gradients in ω_H and ω_0 increase with increasing latitude it may be reasonable to conclude that these gradients will be the dominant factors in off-equatorial interactions. In this case, Figures 7 and 8 will apply to our system. From Figure 7 it can be seen that at low normalized wave frequency, the wave power necesary to sustain a given level of phase trapping efficiency exceeds 17 dB for $\alpha \approx 60°$, when $\psi > 55°$.

For a nonducted wave injected from the ground and propagating within 15° of the magnetic equatorial plane, a value of $\psi > 55°$ would be common for low normalized frequencies [Edgar, 1976]. Thus for low normalized frequencies significantly higher wave power appears necessary in order for a nonducted wave (injected from the ground) to be able to phase trap gyroresonant electrons with a wide range of pitch angle. This situation is much different for the case of mid-frequencies, as shown in Figure 8. It can be seen that for $\alpha \leq 60°$, a given level of phase trapping efficiency can be maintained for all allowable values of ψ with less than a 4 dB power increase. Thus in this frequency range there is a suggestion of a power threshold effect only for resonant particles of high pitch angle. This situation at mid-frequency can change markedly near the magnetic equatorial plane where gradients in ψ are dominant. This is clear from the results of Figure 9 which are based on the approximation relation of (41). For example, for $Y = 2$ and $\psi > 30°$, the threshold value for phase trapping increases from 0.5 to 4mγ when gradients in ψ are considered, an increase of $\sim 18\,dB$.

Consequently, we conclude that near the magnetic equatorial plane gradients in ψ can be quite important in the phase trapping process for nonducted waves. A more precise evaluation of the effects of those gradients must await the development of models that give ψ as a function of magnetic latitude along plasmaspheric magnetic shells for a wide range of injected wave frequency.

4.3. Particle Detrapping

In the past it has been suggested by various authors [e.g., Roux and Pellat, 1978] that VLF emissions may be produced when resonant electrons phase trapped by a ducted whistler mode wave "fall out" of the trap. The fact that $\omega_\tau(\alpha) \Rightarrow 0$ for certain values of α when $\psi > 0$ raises interesting possibilities for terminating the trapping of energetic electrons in gyroresonance with a nonducted wave at points well within the wave structure. For instance, trapped particles will undergo continuous pitch angle changes while within the wave structure (see (47)), and these changes can move the particle pitch angle toward points where $\omega_\tau(\alpha) \Rightarrow 0$. Since the coefficients $\alpha_{1,2}$ in (24c) are independent of wave amplitude, the condition $\omega_\tau \Rightarrow 0$ is independent of B_w also. In this case the particles can be lost from the trap even though the wave amplitude can be quite high.

Referring to Figure 5, it can be seen that large changes in α are not generally necessary for detrapping to occur. For instance, for $\psi = 60°$, trapped particles initially at $\alpha = 65°$ with $\omega_\tau' \approx 0.3$ need undergo only a 3° decrease in α to reach a point where ω_τ has decreased by a factor of 3, and only a 4° decrease to reach the point where $\omega_\tau \Rightarrow 0$.

It is clear then that in the case of nonducted waves, the detrapping of gyroresonant electrons can proceed in a way that is a natural consequence of the dynamics of the interaction.

4.4. Higher-Order Phase Trapping

Equation (48) gives the phase trapping equation for general harmonic cyclotron resonance. From (34c) it can be seen that $h_m \sim v_{Rm}$, where v_{Rm} is given in (47). For large m, $v_{Rm} \sim m$ and consequently, $h_m \sim m$. Thus it requires larger amplitude waves to satisfy a condition like (30) for higher harmonic cyclotron resonance interactions. Consequently the phase trapping mechanism is most efficient for the $m = \pm 1$ resonances (fundamental gyroresonance and anomalous gyroresonance).

4.5. Concluding Remarks

Experimental data indicate that ducted signals from the low-power ($\sim 1\,kW$ radiated power) Siple Station VLF transmitter often trigger VLF emissions as these signals propagate along the duct into the northern hemisphere [Helliwell and Katsufrakis, 1974]. However, satellite observations [Bell et al., 1981; Bell et al., 1983] near the magnetic equatorial plane on L shells between 2-5 and at longitudes within ±15° of the Siple meridian indicate that although nonducted signals from the Siple transmitter are commonly present in this region, they are rarely associated with triggered VLF emissions.

On the other hand, nonducted signals from the higher-powered (10 kW) Omega VLF transmitter in North Dakota are commonly observed to trigger VLF emissions on L shells in the range 2–3.5 [*Bell et al.*, 1981].

These data can be interpreted as showing that for nonducted waves an amplitude threshold exists below which VLF emissions cannot be triggered. Furthermore, this threshold value must be higher than the amplitude threshold that exists for ducted waves.

The results of the present paper agree with this interpretation of the experimental data in that we find that higher values of wave amplitude are necessary to produce a given level of phase trapping efficiency as the wave normal increases from $\psi = 0$. One important factor that we have not been able to evaluate fully here is the variation of ψ along $\mathbf{B}_0$. This quantity must be determined before an accurate estimate of the phase trapping threshold amplitude for nonducted waves can be given.

A detailed ray tracing study is presently underway to calculate the variation of ψ along various magnetic shells for waves injected from ground-based transmitters, using commonly accepted models of the magnetosphere. Results of this work will be reported in the near future.

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REFERENCES

Alfven, H., *Cosmical Electrodynamics*, Oxford University Press, London, 1950.

Anderson, R. R., and K. Maeda, VLF emissions associated with enhanced magnetospheric electrons, *J. Geophys. Res., 82,* 135, 1977.

Angerami, J. J., and J. O. Thomas, Studies of planetary atmospheres, 1, The distribution of electrons and ions in the earth's exopshere, *J. Geophys. Res., 69,* 4537, 1964.

Ashour-Abdalla, M., Amplification of whistler waves in the magnetosphere, *Planet. Space Sci., 20,* 639, 1972.

Bell, T. F., Nonlinear Alfven waves in a Vlasov plasma, *Phys. Fluids, 8,* 1829, 1965.

Bell, T. F., and U. S. Inan, Transient nonlinear pitch angle scattering of energetic electrons by coherent VLF wave packets in the magnetosphere, *J. Geophys. Res., 86,* 9047, 1981.

Bell, T. F., U. S. Inan, and R. A. Helliwell, Nonducted coherent VLF waves and associated triggered emissions observed on the ISEE-1 satellite, *J. Geophys. Res., 86,* 4649, 1981.

Bell, T. F., U. S. Inan, I. Kimura, H. Matsumoto, T. Mukai, and K. Hashimoto, EXOS-B/Siple VLF wave-particle interaction experiments. 2. Transmitter signals and associated emissions, *J. Geophys. Res., 88,* 295, 1983.

Bud'ko, N. I., V. I. Karpman, and O. A. Pokshtelov, Nonlinear theory of the monochormatic circularly polarized VLF and ULF waves in the magnetosphere, *Cosmic Electrodyn., 3,* 147, 1972.

Dowden, R. L., A. C. McKey, L. E. S. Amon, H. C. Koons, and M. H. Dazey, Linear and non-linear amplification in the magnetosphere during a 6.6 kHz transmission, *J. Geophys. Res., 83,* 169, 1978.

Dysthe, K. B., Some studies of triggered whistler emissions, *J. Geophys. Res., 76,* 6915, 1971.

Edgar, B. C.,The theory of VLF doppler signatures and their relation to magnetospheric density structure, *J. Geophys. Res., 81,* 3327, 1976.

Goldstein, H., *Classical Mechanics*, Addison-Wesley, Reading, Mass, 1953.

Helliwell, R. A., *Whistlers and Related Ionospheric Phenomena,* Stanford University Press, Stanford, Calif., 1965.

Helliwell, R. A., A theory of discrete VLF wave emissions fom the magnetosphere, *J. Geophys. Res., 72,* 4773, 1967.

Helliwell, R. A., and T. L. Crystal, A feedback model of cyclotron interaction between whistler-mode waves and energetic electrons in the magnetosphere, *J. Geophys. Res., 78,* 7357, 1973.

Helliwell, R. A., and U. S. Inan, VLF wave growth and discrete emission triggering in the magnetosphere: A feedback model, *J. Geophys. Res., 87,*4427, 1982.

Helliwell, R. A., and J. P. Katsufrakis, VLF wave-injection experiments into the magnetosphere from Siple Station, Antarctica, *J. Geophys. Res., 79,* 2571, 1974.

Inan, U. S., and S. Tkalcevic, Nonlinear equations of motion for Landau response interactions with a whistler mode wave, *J. Geophys. Res., 87,* 2363, 1982.

Inan, U. S., T. F. Bell, and D. L. Carpenter, and R. R. Anderson, Explorer 45 and Imp 6 observations in the magnetosphere of injected waves from the Siple Station VLF transmitter, *J. Geophys. Res., 82(7),* 1177, 1977.

Inan, U. S., T. F. Bell, and R. A. Helliwell, Nonlinear pitch angle scattering of energetic electrons by coherent VLF waves in the magnetosphere, *J. Geophys. Res., 83,* 3235, 1978.

Karpman, V. I., Ja. N. Istomin, and D. R. Shklyar, Nonlinear frequency shift and self-modulation of the quasi-monochromatic whistlers in the inhomogeneous plasma (magnetosphere), *Planet. Space Sci., 22,* 859, 1974a.

Karpman, V. I., Ja. N. Istomin, and D. R. Shklyar, Nonlinear theory of a quasi-monochromatic whistler mode wave packet in inhomogeneous plasma, *Plasma Phys., 16,* 685, 1974b.

Karpman, V. I., Ja. N. Istomin, and D. R. Shklyar, Effects of nonlinear interaction of monochromatic waves with resonant particles in the inhomogeneous plasma, *Physica Scripta, 11,* 278, 1975.

Kennel, C. F., and H. E. Petschek, Limit on stably trapped particle fluxes, *J. Geophys. Res., 71,* 1, 1966.

Kimura, I., H. Matsumoto, T. Mukai, K. Hashimoto, T. F. Bell, U. S. Inan, R. A. Helliwell, and J. P. Katsufrakis, EXOS-B/Siple Station VLF wave-particle interaction experiments: 1. General description and wave-particle correlations, *J. Geophys. Res., 88,* 292, 1983.

Koons, H. C., M. H. Dazey, R. L. Dowden, and L. E. S. Amon, A controlled VLF phase-reversal experiment in the magnetosphere, *J. Geophys.Res., 81,* 5536, 1976.

Liemohn, H. B., Cyclotron-resonance amplification of VLF and ULF whistlers, *J. Geophys. Res., 72,* 39, 1967.

Matsumoto, H., and I. Kimura, Linear and nonlinear cyclotron instability and VLF emissions in the magnetosphere, *Planet. Space Sci., 19,* 567, 1971.

Matsumoto, H., and Y. Omura, Cluster- and channel-effect phase bunching by whistler waves in a nonuniform geomagnetic field, *J. Geophys. Res., 86,* 779, 1981.

Matsumoto, H., K. Hashimoto, and I. Kimura, Dependence of coherent nonlinear whistler interaction on wave amplitude, *J. Geophys. Res., 85,* 644, 1980.

McPherson, D. A., H. C. Koons, M. H. Dazey, R. L. Dowden, L. E. S. Amon, and N. R. Thomson, Conjugate magnetospheric transmissions at VLF from Alaska to New Zealand, *J. Geophys. Res., 79,* 1555, 1974.

Minorsky, N., *Nonlinear Oscillations,* Van Nostrand, Princeton, N.J., 1962.

Nunn, D., A theory of VLF emissions, *Planet. Space Sci., 19,* 1141, 1971.

Nunn, D., A self-consistent theory of triggered VLF emissions, *Planet. Space Sci., 22,* 349, 1974.

Omura, Y., and H. Matsumoto, Computer simulation of basic processes of coherent whistler wave-particle interactions in the magnetosphere, *J. Geophys. Res., 87,* 4435, 1982.

Palmadesso, G., and G. Schmidt, Collisionless damping of a large amplitude whistler wave, *Phys. Fluids, 14,* 1411, 1971.

Palmadesso, G., and G. Schmidt, Stability of a steady, large amplitude whistler wave, *Phys. Fluids, 15,* 485, 1972.

Park, C. G., D. L. Carpenter, and D. B. Wiggin, Electron density in the plasmasphere: Whistler data on solar cycle, annual, and diurnal variations, *J. Geophys. Res., 83,* 3137, 1978.

Roux, A., and R. Pellat, A study of triggered emissions, in *Magnetospheric Particles and Fields,* edited by B. M. McCormac, p. 209, D. Reidel, Hingham, Mass., 1976.

Roux, A., and R. Pellat, A theory of triggered emissions, *J. Geophys. Res., 83,* 1433, 1978.

Stix, T. H., *The Theory of Plasma Waves,* McGraw-Hill Book Company, New York, 1962.

Vomvoridis, J. L., and J. Denavit, Test particle correlation by a whistler wave in a nonuniform magnetic field, *Phys. Fluids, 22,* 367, 1979.

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