the beat note continues. The integrator output therefore stays very close to zero as long as the loop is out of lock.

The frequency rate decreases once the deviation has passed through zero. At some point the rate will become small enough that the loop could hold track—if it were already locked. But the loop will not be locked at that point. The VCO frequency is determined by the integrator, which stayed near zero when there was little dc output from the phase detector, while the signal frequency has followed the modulation. There is now a substantial frequency discrepancy between VCO and signal and that discrepancy increases, at least up to the peak of the modulation cycle. That frequency difference is large enough to prevent rapid relocking.

Past the modulation peak, the modulation frequency turns around and starts moving back towards that of the VCO. If it were to move slowly enough, the loop would lock up when the two frequencies came into coincidence. But, because the modulation is sinusoidal, the rate of change of frequency increases so that, when the two frequencies coincide, the rate of change is excessive and the loop is incapable of locking.

And so it goes. Whenever the frequency rate is small enough, the frequency difference is too large for rapid locking and whenever the frequency difference is small, the frequency rate is too large. As a result, the loop can never lock at any point in the cycle. That is why one observes a complete breakup of tracking in the second order PLL, rather than the isolated spikes seen in the first order loop.

If an nth-order PLL is subjected to a frequency modulated signal (whose modulation frequency is small compared to the loop bandwidth) then the peak phase error occurs at the peak of the $(n-1)$th derivative of the modulation. If the modulation is sinusoidal and n is odd then the error peak will coincide with the modulation peak. That raises the possibility and odd-order PLLs will exhibit isolated spikes as seen in the first order loop, rather than the breakup seen in the second order PLL.

REFERENCES

(1) F. M. Gardner and J. F. Heck, "Angle Modulation Limits of a Noise-Free Phase Lock Loop." this issue, pp. 1129-1136.
(2) F. M. Gardner, *Phaselock Techniques,* Wiley: New York, 1966.

A Novel Approach for Finding the Spectrum of Periodically Modulated FM Carriers

DAVRAS YAVUZ AND UMRAN SAVAŞ İNAN

*Abstract—*A new technique for finding the frequency spectrum of a carrier frequency modulated by a periodic signal is presented. The technique is based on the frequency modulation (FM) equation due to Hess, which in the frequency domain is a double convolution equation.

The technique reduces the FM spectrum problem to the solution of a set of linear equations in which the FM side-band amplitudes are the unknowns. The results obtained through this new technique can, in most cases, be obtained by other means. However, the technique is simple and as a result its implementation on a standard computer is both simple and fast. Furthermore, it is especially suitable when the modulating signal is in the form of a Fourier series with a large number of terms or requires a large number of terms to approximate it. Numerical results obtained using the technique for a large number of complex modulating signals are presented.

INTRODUCTION

The problem of finding the frequency spectrum of a sinusoidal carrier whose frequency is modulated by a periodic signal is a relatively old one. The results for sinusoidal and square wave modulation are well known. For complex modulating signals, one generally starts by seeking an analytic solution for the integral expressions for the Fourier coefficients. Another common approach is to search for a series expansion for the related complex exponential. In general the problem is a difficult one and few cases in which analytic solutions are possible are known. When an analytic solution does not seem possible, one resorts to various numerical techniques for evaluating the Fourier coefficients of the modulated carrier.

In general, the technique of solution for each modulating signal ends up being specifically useful only for that particular signal.

In this paper we present a general technique which is suitable for all types of periodic modulating signals.

THE FM EQUATION

The FM equation [1, 2] is a second order, linear, homogeneous equation, which in integro-differential form is given by

$$\omega_i(t) \int^t \omega_i(u)v(u)\, du + \frac{dv(t)}{dt} = 0. \tag{1}$$

This equation may also be written in differential form as

$$v(t) - \frac{\dot{v}(t)\omega_i(t)}{\omega_i^3(t)} + \frac{\ddot{v}(t)}{\omega_i^2(t)} = 0 \tag{2}$$

where $\omega_i(t) > 0$ and $v(t) = dv(t)/dt$ etc. By direct substitution into (1) or (2) it is easily verified that two linearly independent solutions to the FM equation are

$$v_1(t) = \cos \int^t \omega_i(u)\, du, \qquad v_2(t) = \sin \int^t \omega_i(u)\, du. \tag{3}$$

Thus the general solution to the FM equation may be written either as

$$v(t) = A\cos \int^t \omega_i(u)\, du + B\sin \int^t \omega_i(u)\, du \tag{4}$$

or as

$$v(t) = C \cos \left[\int^t \omega_i(u) \, du + \theta \right] \tag{5}$$

where A, B, C, θ are constants.

Clearly, (4) or (5) are the well known expressions for a sinusoidal FM carrier with an instantaneous radian frequency equal to $\omega_i(t)$, which will normally be of the form

$$\omega_i(t) = \omega_0 + \Delta\omega f(t) \tag{6}$$

where ω_0 is the unmodulated carrier radian frequency, $\Delta\omega$ the peak deviation and $f(t)$ the modulating signal. The previously made assumption $\omega_i(t) > 0$, $\forall t$ clearly is the same as assuming that the deviation is always less than the carrier frequency, which of course is the case in all realistic situations.

To obtain information about the spectral properties of the FM signal $v(t)$, we transform eq. (1) into the frequency domain and obtain the following second order convolution equation

$$V(\omega) = \frac{1}{(2\pi)^2} \frac{1}{\omega} \left\{ \Omega_i(\omega) * \left[\frac{\Omega_i(\omega) * V(\omega)}{\omega} \right] \right\},$$

$$\omega \neq 0. \tag{7}$$

Here $*$ denotes convolution and $V(\omega)$ and $\Omega_i(\omega)$ are the Fourier transforms of $v(t)$ and $\omega_i(t)$ respectively. Equation (7), which relates the spectrum of the instantaneous frequency to the spectrum of the FM signal, can be used to give a great deal of information concerning FM spectra. For example, it may be used to prove in a very simple yet elegant way that no FM process can be band-limited [3, 4].

In the remainder of this paper we will be concerned with the solution of eq. (7).

FM SPECTRA VIA THE FM EQUATION

The double convolution equation (7) does not appear to belong to any of the various types of equations extensively studied in mathematics literature. The technique we have developed for the solution of (7), when the modulating signal $f(t)$ is periodic, will be presented in the following [5].

Without any loss of generality, we assume that the modulating signal $f(t)$ in (6) is periodic with period 2π s.

Since $f(t)$ is periodic, with fundamental radian frequency normalized to 1 rps,

$$f(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{jnt} \qquad \alpha_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-jnt} \, dt \tag{8}$$

$$F(\omega) = \mathcal{F}\{f(t)\} = 2\pi \sum_{n=-\infty}^{\infty} \alpha_n \delta(\omega - n). \tag{9}$$

Taking the Fourier transform of (6) and substituting (9)

$$\Omega_i(\omega) = 2\pi \left[\omega_0 \delta(\omega) + \Delta\omega \sum_{n=-\infty}^{\infty} \alpha_n \delta(\omega - n) \right]. \tag{10}$$

Substituting (10) into (7), we obtain after some simplification the following infinite order difference equation with varying

coefficients

$$\left(\frac{\omega_0^2 - \omega^2}{\omega} \right) V(\omega) + \omega_0 \Delta\omega \sum_{n=-\infty}^{\infty} \alpha_n \left(\frac{2\omega - n}{\omega^2 - \omega n} \right)$$

$$\cdot V(\omega - n) + (\Delta\omega)^2 \sum_{n=-\infty}^{\infty} \frac{\alpha_n}{\omega - n}$$

$$\cdot \sum \alpha_m V(\omega - n - m) = 0. \tag{11}$$

Since the sine and cosine of periodic arguments is also periodic the FM spectrum $V(\omega)$ must normally be[1] of the form

$$V(\omega) = \frac{1}{2} \sum_{p=-\infty}^{\infty} [A_p \delta(\omega - \omega_0 - p) + A_p \delta(\omega + \omega_0 + p)]$$

$$= V^+(\omega) + V^-(\omega) \tag{12}$$

where $V^+(\omega)$ and $V^-(\omega)$ are simply the positive and negative frequency portions of the spectrum. Replacing (12) into (11) we obtain

$$\frac{\omega_0^2 - \omega^2}{\omega} \sum_{p=-\infty}^{\infty} A_p [\delta(\omega - \omega_0 - p) + \delta(\omega + \omega_0 + p)]$$

$$+ \omega_0 \Delta\omega \sum_{n=-\infty}^{\infty} \alpha_n \left(\frac{2\omega - n}{\omega^2 - \omega n} \right)$$

$$\cdot \sum_{p=-\infty}^{\infty} A_p [\delta(\omega - \omega_0 - p - n) + \delta(\omega + \omega_0 + p - n)]$$

$$+ (\Delta\omega)^2 \sum_{n=-\infty}^{\infty} \frac{\alpha_n}{\omega - n} \sum_{m=-\infty}^{\infty} \alpha_m$$

$$\cdot \sum_{p=-\infty}^{\infty} A_p [\delta(\omega - \omega_0 - p - n - m)$$

$$+ \delta(\omega + \omega_0 + p - n - m)] = 0. \tag{13}$$

For the equality in (13) to be true, the coefficients of each of the impulse terms in (13) must separately be equal to zero. These coefficients are linear equations in terms of A_p's, with ω_0 a parameter.

Recall that we had assumed the modulating signal to have an infinite number of terms in its Fourier expansion, eq. (8). Assume now that $f(t)$ can be represented by the first N MAX terms in its Fourier series representation. If $f(t)$ is expressible in terms of a finite number of terms in its Fourier series, clearly this assumption does not imply an approximation. Otherwise, the approximation error may be reduced to negligible levels by a suitably large choice of N MAX.

Furthermore, we assume that a finite number of terms in the spectrum of the modulated carrier are significant. That is,

---

[1] This is not strictly true when the carrier frequency and modulating signal fundamental frequency are close to each other and are not related by an integer factor. Such cases must be treated separately.

we assume that A_p's for $|P| > P$ MAX are of negligible magnitude. By choosing P MAX suitably large, one may obtain any degree of accuracy for side-band amplitudes. Hence this assumption too is not restrictive.

As a third assumption we assume that $\omega_0 > P$ MAX, which clearly is also non-restrictive. Another point is that, since we assume ω_0 to be sufficiently large so that the positive and negative frequency spectra do not over lap, the $V^+(\omega)$ and $V^-(\omega)$ terms in (12) must each satisfy eq. (11). Thus we need only consider $V^+(\omega)$.

With these assumptions, eq. (13) may now be written as

$$\frac{\omega_0^2 - \omega^2}{\omega} \sum_{p=-P\,MAX}^{P\,MAX} A_p \delta(\omega - \omega_0 - p)$$

$$+ \omega_0 \Delta\omega \sum_{n=-N\,MAX}^{N\,MAX} \alpha_n \left(\frac{2\omega - n}{\omega^2 - \omega n}\right)$$

$$\cdot \sum_{p=-P\,MAX}^{P\,MAX} A_p \delta(\omega - \omega_0 - p - n)$$

$$+ (\Delta\omega)^2 \sum_{n=-N\,MAX}^{N\,MAX} \frac{\alpha_n}{\omega - n} \sum_{m=-N\,MAX}^{N\,MAX} \alpha_m$$

$$\cdot \sum_{p=-P\,MAX}^{P\,MAX} A_p \delta(\omega - \omega_0 - p - n - m) = 0. \qquad (14)$$

When we equate the coefficients of the impulse terms in (14) to zero, we obtain $2P$ MAX $+$ 1 linear, homogeneous equations for the $2P$ MAX $+$ 1 unknowns $A_{-P\,MAX}$, $A_{-P\,MAX+1}$, $\cdots$, $A_0, A_1, \cdots, A_{P\,MAX} = \{A_i\}$. It is well known that, with the exception of certain pathological cases, the power of an FM carrier is independent of the modulating parameters such as the deviation and the modulating signal. This gives the non-homogeneous equation[2]

$$\sum_{p=-P\,MAX}^{P\,MAX} A_p^2 = 1 \qquad (15)$$

for the unknowns $\{A_i\}$.

Thus, the equations equating impulse term coefficients to zero from (14) and eq. (15) may be solved simultaneously to give the $\{A_i\}$. The parameter ω_0 is simply assigned a suitably large numerical value.

A computer program in FORTRAN IV which is inputted with β the modulation index (deviation/fundamental modulation frequency), ω_0, N MAX, functional expression for α_n (or values of α_n), P MAX and uses Gauss elimination for the solution of the equations, can be written with about eighty statements[3]. The program finds the side-band amplitudes for most modulating signals in the order of tens of seconds for each β with an IBM 360/40. When a program is written for the

[2] Strictly speaking, eq. (15) is an approximation since PMAX is finite. However, as stated in connection with the assumptions, it is possible to reduce the resultant numerical errors to any desired level by suitably large choices of PMAX.

[3] Program information will be supplied to interested readers if they request so from D. Yavuz.

solution of the equations in A_p a convenient way to check it is with sinusoidal or square wave modulation. Side-band amplitudes are easily calculated in terms of Bessel functions and are readily available in many sources [6, 7]. Those for square-wave modulation may be found in [8]. Another check is to change ω_0 and obtain the same results, assuming of course that ω_0 is always kept suitably large.

As stated previously, P MAX may be chosen sufficiently large to give any accuracy desired, provided numerical limitations due to the computer word size are not encountered. However, in practice, knowing the FM spectrum side-band amplitudes to an accuracy of about $\pm 10^{-3}$ (unmodulated carrier amplitude $= 1.0$) is generally sufficient. It is thus convenient to define a Carson's rule multiplying factor 'k' and choose P MAX according to P MAX $= k(\beta + 1)$. As a result of literally hundreds of computer runs, we have found that $k = 2, 3$, or 4 is sufficient for almost all possible modulating waveforms that could be of interest. As seen in Fig. 1, $k = 2$ is suitable for relatively smooth modulating signals. For signals with fast edges, $k = 3$ or 4 should be used.

Signals [1] and [13] are program test signals for which the spectrum is readily found. Signal [12] has a power spectrum approximating typical speech signals. All fundamental frequencies have been normalized to 2π s in the computer program. The spectrum results are given in terms of $\omega_m = 2\pi f_m$, f_m = fundamental frequency of the modulating signal $f(t)$ in Hz.

To illustrate some of the results obtained with the technique introduced in this paper, in Table 1 we give the side-band amplitudes for the 18 signals shown in Fig. 1. For $\beta = 1$ five side-bands, for $\beta = 5$ thirteen side-bands on each side of the carrier ω_0 are given. FM spectrum information for most of these signals is not available elsewhere. Finally, in Fig. 2 we give the FM spectra for some of the modulating signals in graphical form.

CONCLUSION

A new technique for finding the spectrum of a sinusoidal carrier frequency modulated by any periodic signal is presented. The technique reduces the FM spectrum problem to the simultaneous solution of linear equations in which side-band amplitudes are the unknowns. As such, it is easily and speedily implemented with a simple program on any standard computer. The modulating signal is entered into the program in terms of its Fourier coefficients, which is usually both desirable and convenient.

Although this paper has presented the technique based on the frequency domain version of the FM equation, it is also possible to arrive at the same method of solution of the FM spectrum problem by using time domain techniques, e.g. writing $v(t)$ and $\omega_i(t)$ as series of complex exponentials. However, this approach is somewhat more involved.

An interesting area of work is the problem of solving (7) for continuous modulating spectra, i.e. for deterministic finite energy signals. Although such results would probably be mainly of theoretical interest, they could lead to interesting insights. For example, it might be possible to find the exact spectrum when the modulating signal has a spectrum of the form $e^{-k|\omega|}$ where k is a positive real constant through an iterative technique using (7).

[1] Square wave

PMAX=3[β+1]
k=3
NMAX=PMAX+1

$\alpha_n = \sin[n\pi/2]/[n\pi/2]$

[2] Wide pulse

PMAX=4[β+1]
k=4
NMAX=PMAX+1

$\alpha_n = 0.5\sin[n\pi(0.25)]/[n\pi(0.25)]$

[3] Narrow pulse

PMAX=4[β+1]
k=4
NMAX=PMAX+1

$\alpha_n = 0.2\sin[n\pi(0.1)]/[n\pi(0.1)]$

[4] Raised cosine

PMAX=2[β+1]
k=2
NMAX=PMAX+1

$\alpha_n = 4\sin^2(n\pi/2)/\pi n(4-n^2)$, $\alpha_2 = 0$

[5] Half wave cosine

PMAX=3[β+1]
k=3
NMAX=PMAX+1

$\alpha_n = -2\cos(n\pi/2)/(\pi(n^2-1))$, $\alpha_1=0.25$

[6] Full wave cosine

PMAX=4[β+1]
k=4
NMAX=PMAX+1

$\alpha_n = -4\cos(n\pi/2)/(\pi(n^2-1))$, $\alpha_1 = 0$

[7] $f(t) = 0.1\cos t + 0.5\cos 2t + \cos 3t$

$F(\omega)$
$A_3 = 1.0$
$A_2 = 0.5$
$A_1 = 0.1$
PMAX=2[β+1]+2
k=2
NMAX=3

$\alpha_n = A_n/2$

[8] $f(t) = 0.5\cos t + \cos 2t + 0.1\cos 3t$

$F(\omega)$
1.0
0.5
0.1
PMAX=2[β+1]+2
k=2
NMAX=3

[9] $f(t) = \cos t + \cos 2t$

$F(\omega)$
1.0 1.0
PMAX=3[β+1]
k=3
NMAX=2

[10] $f(t) = \cos t + \cos 3t$

$F(\omega)$
1.0 1.0
PMAX=3[β+1]
k=3
NMAX=3

[11] Flat spectrum

$F(\omega)$
1.0 1.0 1.0 1.0
PMAX=4[β+1]
k=4
NMAX=8

[12] Speech spectrum

$F(\omega)$
1.0
0.79
0.63
0.26
0.1 0.079 0.007
PMAX=4[β+1]
k=4
NMAX=16

[13] $f(t) = \cos t$

$F(\omega)$
1.0
PMAX=2[β+1]
k=2
NMAX=1

[14] $f(t) = \cos t + 0.1\cos 2t$

$F(\omega)$
1.0
0.1
PMAX=2[β+1]
k=2
NMAX=2

[15] $f(t) = \cos t + 0.5\cos 2t$

$F(\omega)$
1.0
0.5
PMAX=2[β+1]
k=2
NMAX=2

[16] $f(t) = \cos t + 0.1\cos 3t$

$F(\omega)$
1.0
0.1
PMAX=2[β+1]
k=2
NMAX=3

[17] $f(t) = \cos t + 0.5\cos 3t$

$F(\omega)$
1.0
0.5
PMAX=2[β+1]
k=2
NMAX=3

[18] $f(t) = \cos t + 0.5\cos 2t + 0.1\cos 3t$

$F(\omega)$
1.0
0.5
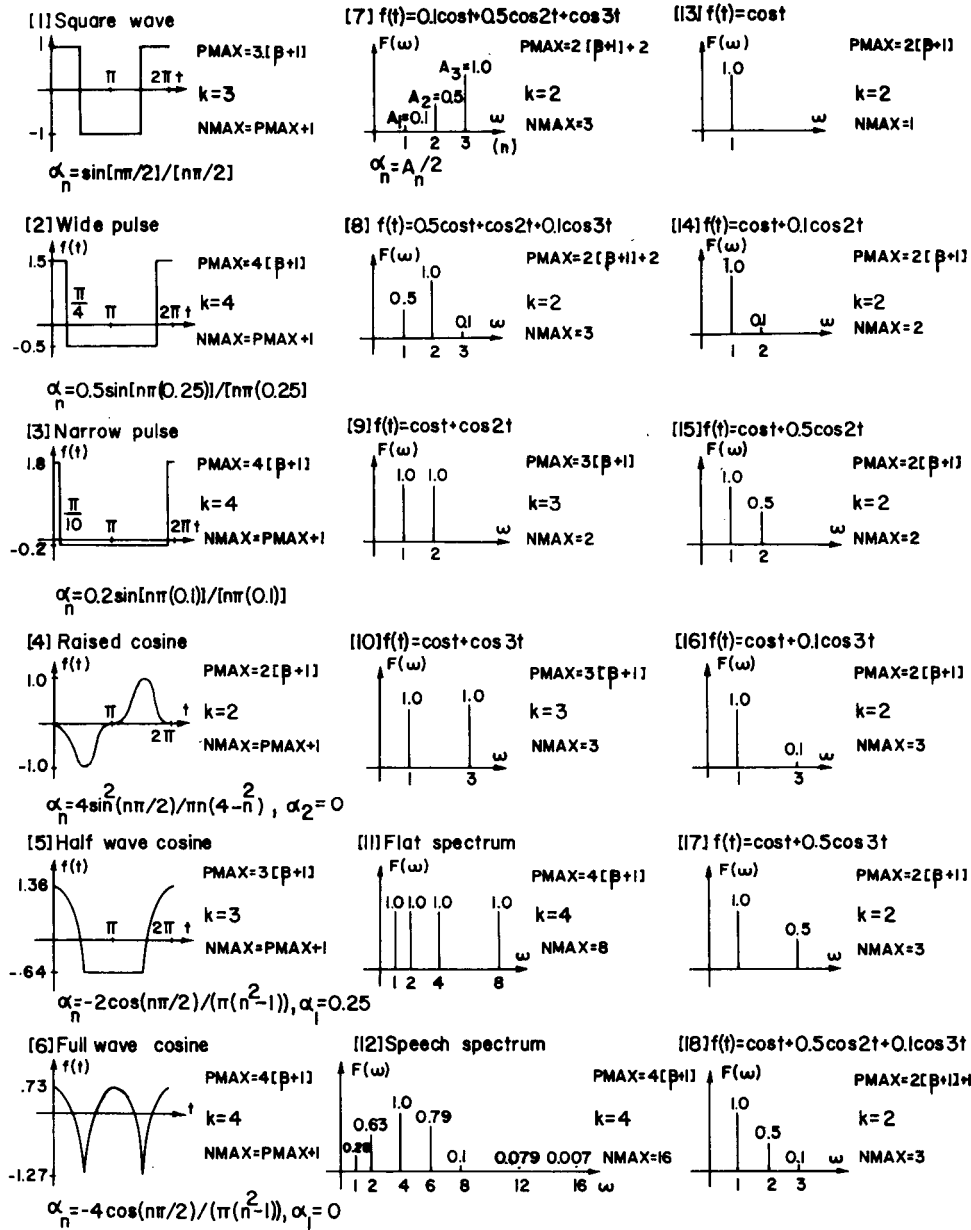0.1
PMAX=2[β+1]+1
k=2
NMAX=3

Figure 1. The 18 modulating signals for which the frequency spectrum
has been calculated. Signals [1] to [6] are shown in the time-
domain, the remainder in the frequency domain. [1] and [13] are
program test signals.

TABLE 1
SOME NUMERICAL RESULTS GIVING THE MAGNITUDE OF THE SIDE-BANDS FOR THE SPECTRUM OF A SINUSOIDAL CARRIER, FREQUENCY MODULATED BY THE 18 SIGNALS OF FIG. 1

location of side-band	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\omega_0 - 5$		.020	.016		.004		.020	.004	.010	.012	.052	.010						.001
$\omega_0 - 4$	.042	.030	.027	.009	.011	.041	.017	.026	.002	.075	.100	.118	.002		.006	.009	.038	
$\omega_0 - 3$		.021	.047	.014	.026		.154	.048	.101	.144	.051	.008	.019	.008	.038	.032	.082	.026
$\omega_0 - 2$	.212	.046	.086	.060	.067	.202	.126	.200	.073	.039	.120	.167	.114	.095	.019	.107	.076	.013
$\omega_0 - 1$	.499	.470	.218	.338	.265		.072	.290	.523	.446	.523	.158	.440	.451	.490	.441	.445	.493
carrier ω_0	.637	.784	.947	.873	.927	.954	.955	.880	.718	.738	.692	.936	.765	.764	.753	.764	.757	.752
$\omega_0 + 1$	.499	.324	.164	.338	.213		.020	.164	.301	.446	.316	.116	.440	.428	.376	.441	.445	.375
$\omega_0 + 2$	.212	.194	.090	.060	.130	.212	.112	.252	.296	.039	.288	.132	.114	.133	.208	.107	.076	.200
$\omega_0 + 3$		.111	.060	.014	.029		.167	.068	.111	.144	.055		.019	.030	.070	.032	.082	.082
$\omega_0 + 4$	.042	.052	.043	.009	.001		.012	.039	.053	.075	.035	.120	.002	.005	.023	.009	.038	.028
$\omega_0 + 5$		.012	.032		.001		.019	.011	.017	.012	.064	.005						.009

$\beta = 1$

location of side-band	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\omega_0 - 13$		.005	.006		.001		.014	.001	.001	.041	.031	.041						
$\omega_0 - 12$	.026	.006	.003		.003	.021	.009	.003	.006	.064		.051 .					.014	
$\omega_0 - 11$		.018			.002		.020	.005		.092	.075	.030				.002	.027	
$\omega_0 - 10$	.042	.022	.006		.008	.044	.064	.015	.028	.111	.035	.001	.001			.007	.047	
$\omega_0 - 9$		.011	.013	.001	.004		.013	.022	.021	.157	.177	.139	.005		.005	.017	.080	
$\omega_0 - 8$	.081	.014	.021	.003	.019	.095	.093	.059	.086	.203	.045	.040	.018	.005	.009	.040	.122	.006
$\omega_0 - 7$		.044	.031	.009	.009		.195	.081	.134	.156	.240	.014	.053	.023	.019	.084	.165	.002
$\omega_0 - 6$	.288	.061	.042	.018	.053	.200	.052	.167	.135	.185	.241	.140	.131	.083	.090	.159	.227	.039
$\omega_0 - 5$	.499	.038	.053	.021	.023		.237	.230	.444	.245	.312	.202	.261	.215	.059	.262	.260	.030
$\omega_0 - 4$	.354	.070	.065	.008	.167	.372	.362	.308	.031	.063	.184	.147	.391	.394	.269	.357	.182	.251
$\omega_0 - 3$		.560	.077	.108	.114		.278	.431	.506	.113	.229	.375	.364	.435	.613	.335	.159	.618
$\omega_0 - 2$	.152	.619	.089	.289	.451	.489	.186	.245	.288	.165	.209	.376	.046	.117	.337	.953	.098	.362
$\omega_0 - 1$		.095	.900	.486	.653		.313	.283	.052	.390	.255	.586	.327	.318	.198	.354	.420	.238
carrier ω_0	.127	.065	.109	.576	.005	.142	.301	.108	.360	.423	.286	.003	.177	.168	.0193	.236	.407	.029
$\omega_0 + 1$		.129	.117	.486	.014		.361	.474	.120	.390	.297	.155	.327	.328	.260	.354	.420	.244
$\omega_0 + 2$	.152	.118	.122	.289	.309	.648	.071	.095	.114	.165	.144	.056	.046	.023	.215	.053	.098	.219
$\omega_0 + 3$		.049	.126	.108	.349		.403	.056	.240	.113	.156	.106	.364	.288	.015	.335	.159	.038
$\omega_0 + 4$	.354	.053	.127	.008	.270	.343	.040	.242	.090	.063	.091	.128	.391	.373	.186	.357	.182	.146
$\omega_0 + 5$	.500	.157	.126	.021	.153		.235	.201	.037	.245	.338	.089	.261	.293	.282	.262	.260	.251
$\omega_0 + 6$	.289	.230	.122	.018	.073	.099	.242	.247	.173	.185	.054	.229	.131	.175	.277	.159	.227	.266
$\omega_0 + 7$		.256	.116	.009	.034		.075	.190	.232	.156	.050	.036	.053	.086	.217	.089	.165	.226
$\omega_0 + 8$	.081	.231	.108	.003	.014	.014	.139	.151	.237	.203	.203	.215	.018	.037	.145	.040	.122	.165
$\omega_0 + 9$		.170	.099	.001	.003		.097	.101	.202	.157	.031	.019	.005	.014	.087	.017	.080	.108
$\omega_0 + 10$	.042	.093	.089			.003	.047	.066	.152	.111	.121	.182	.001	.004	.047	.007	.047	.065
$\omega_0 + 11$		.025	.077				.049	.039	.104	.092	.065	.017		.001	.023	.002	.027	.036
$\omega_0 + 12$	.026	.018	.065		.001		.028	.022	.066	.064	.133	.151			.012		.013	.019
$\omega_0 + 13$		.034	.053				.014	.011	.039	.041	.024	.036						.009

$\beta = 5$
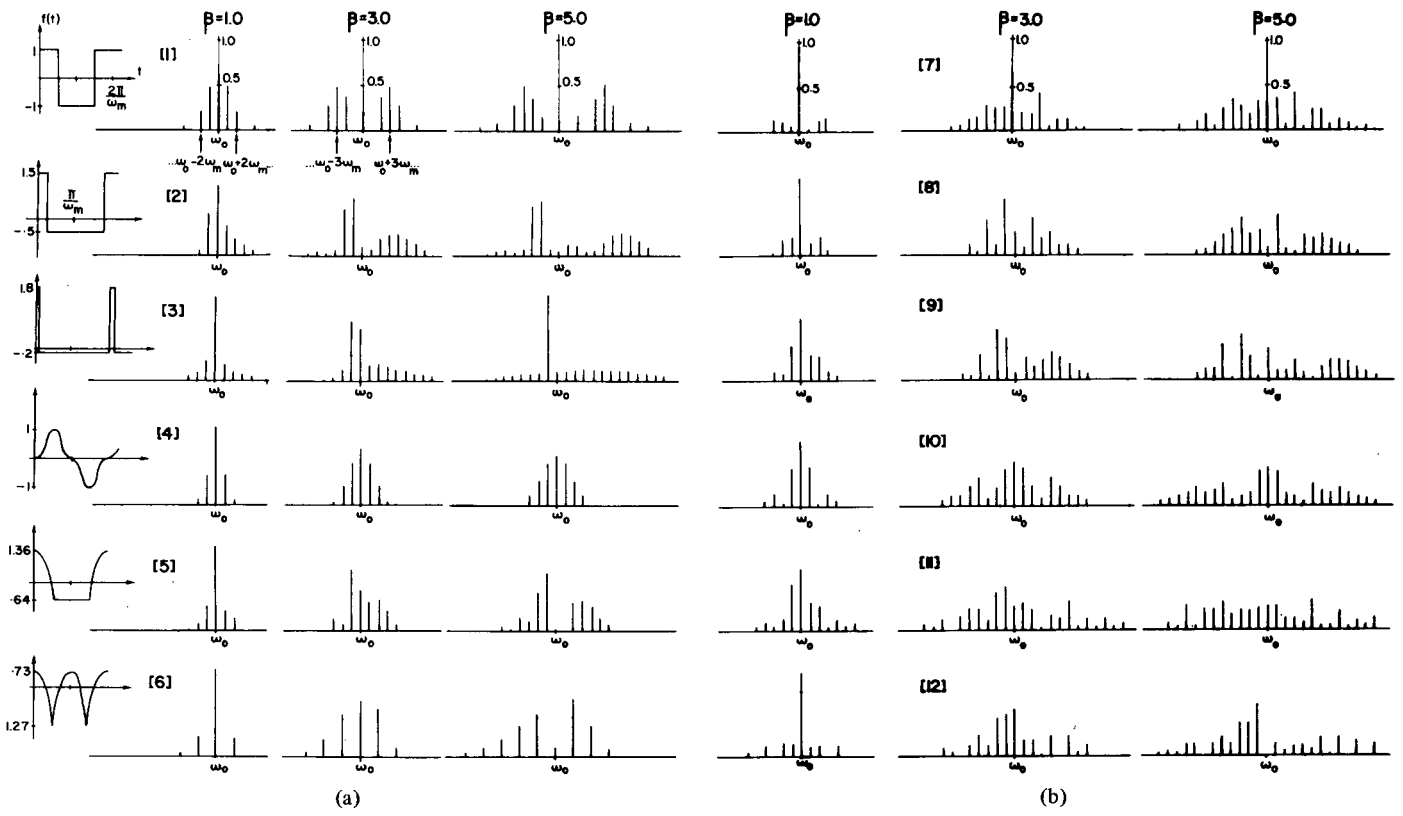
(a)   (b)
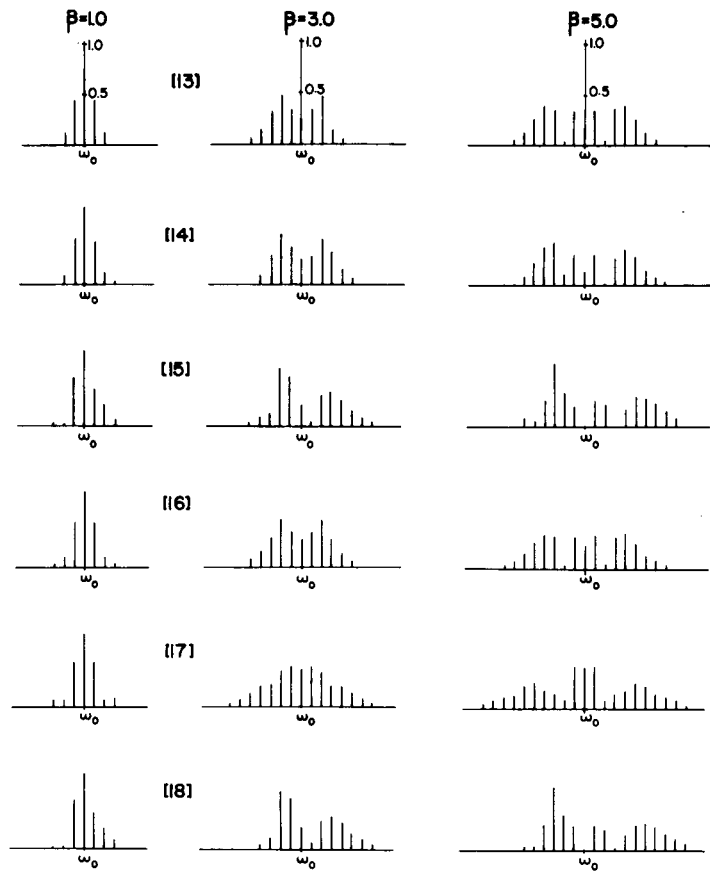
(c)

Figure 2.   Some representative examples of magnitude spectra for β =
1, 3, 5. All horizontal and vertical axis are identical. Vertical axis
is linear.   Side-bands whose magnitude is less than 0.02 are not
indicated (unmodulated carrier amplitude = 1.0).

REFERENCES

[1] D. T. Hess, "FM Differential Equation," (Letter) *Proc. IEEE*, Vol. 54, pp. 1089, August 1966.

[2] K. K. Clarke and D. T. Hess, *Communication Circuits*, Addison-Wesley, 1971.

[3] D. Yavuz, "Frequency Modulation Spectra," Report for the Degree of M.Sc., Polytechnic Institute of Brooklyn, 1967.

[4] D. Yavuz, "A Note on FM Spectra" (Letter) *Proc. IEEE*, Vol. 56, pp. 220-221, Feb. 1968. (see also *Proc. IEEE*, July 1968 pp. 1241).

[5] Umran Savaş İnan, "Angle Modulation Spectra", M.Sc. Thesis, Middle East Technical University, 1973.

[6] M. Schwartz, *Information Transmission, Modulation and Noise*, McGraw-Hill, 1970.

[7] H. Taub and D. L. Schilling, *Principles of Communication Systems*, McGraw-Hill, 1971.

[8] H. S. Black, *Modulation Theory*, D. Van Nostrand, 1953.

Bounds on the Number of Possible Distinct Networks

MARTIN NESENBERGS

Abstract—This note is concerned with the number of all distinct networks, also called connected graphs, or Cayley, or linear graphs that are possible for n given nodes. Since for any practical number of nodes an explicit enumeration appears difficult, if not impossible, we offer an improved lower bound that enables tight approximation for n reasonably large.

I. BACKGROUND

First, a few words on nomenclature. By a network we shall mean a connected graph that consists of nodes (points) and links (lines). The nodes are labeled, so that interchange of communications stations is not feasible without configuring a generally new network. The connected graphs are to be interpreted in the sense of Cayley [1] as simple linear bidirectional graphs. According to this, a tree, a star, or a loop network (graph) has its conventional topology.

It has been known for some time that the number of distinct tree networks is n^{n-2} for $n \geqslant 2$ nodes. An early proof was given by Cayley [1], followed by others [2, 3]. Cayley's standard tree network with n nodes is configured without closed loops and possesses $n - 1$ links.

In communications one encounters other types of networks besides trees, such as the familiar loops, grids, ARPA multiloop types, and so forth. The common feature of all such networks is connectivity and a numerical value for n in the neighborhood of 100 or more. In this note we are concerned with the number, $N(n)$, of all possible and distinct connected networks that can be formed for n nodes. To our knowledge, a simple closed form expression for $N(n)$ is lacking. The following is known.

Since a direct line is possible between any pair of nodes, and there are $n(n - 1)/2$ pairs, the total number of distinct connected *plus disjoint* graphs is $2^{n(n-1)/2}$. For $n \geqslant 2$ there are always some disjoint graphs and, therefore, $N(n) < 2^{n(n-1)/2}$.

Furthermore, it has been shown by graph theorists that the disjoint graphs form a more and more insignificant fraction as n increases, and that asymptotically $N(n) \sim 2^{n(n-1)/2}$ for large n [4–7].

Attempts to find a lower bound for $N(n)$ have also been made. Thus, it has been shown [7, Section 9.4] that

$$N(n) \geqslant (1 - 2n2^{-n/2})2^{n(n-1)/2}. \tag{1}$$

II. THE LOWER BOUND

In this section we present a lower bound on $N(n)$ that is considerably tighter for $n \gg 1$. A lower bound on $N(n)$ is the same as an upper bound on the number of disjoint graphs, namely on $2^{n(n-1)/2} - N(n)$. Each disjoint graph for $n \geqslant 2$ points has at least one cut that partitions that graph into two sub-graphs. These sub-graphs may be permitted to be arbitrarily connected or not. Let the two sub-graphs have j and $n - j$ nodes, respectively, and assume $1 \leqslant j \leqslant n - j \leqslant n - 1$. For each fixed j, there are then at most $\binom{n}{j}$ possible cuts, and for each cut there are as many as $2^{j(j-1)/2}$ times $2^{(n-j)(n-j-1)/2}$ distinct possibilities. For all $n \geqslant 3$, duplication of sub-graphs occurs. Thus, whether the number of nodes is even ($n = 2m$) or odd ($n = 2m + 1$), this type of partition must yield a strict inequality for the number of disjoint graphs:

$$2^{n(n-1)/2} - N(n) < \sum_{j=1}^{m} N(n \mid j), \tag{2}$$

where the partial bound $N(n \mid j)$ is defined for each j as

$$N(n \mid j) = \binom{n}{j} 2^{j(j-1)/2 + (n-j)(n-j-1)/2}. \tag{3}$$

Substitution of (3) into (2) yields a temporary bound on $N(n)$,

$$\frac{N(n)}{2^{n(n-1)/2}} > 1 - \sum_{j=1}^{m} \binom{n}{j} 2^{-j(n-j)}. \tag{4}$$

This lower bound is reasonably tractable for smaller n or with the aid of computing machines. By separately upper-bounding the summed entities in (4), namely $\binom{n}{j}$ and $2^{-j(n-j)}$, one can in fact deduce another bound that is nearly twice as tight as previously quoted in (1).

To obtain an even better bound, return to the term $N(n \mid j)$ defined in (3). The ratio of such successive terms is

$$\frac{N(n \mid j + 1)}{N(n \mid j)} = \frac{2^{(j+1) - \log_2(j+1)}}{2^{(n-j) - \log_2(n-j)}}, \tag{5}$$

which happens to be less than or equal to unity for all j in the interval $1 \leqslant j \leqslant m - 1$. One way to show this is to note that