On Input Impedance of an Arbitrarily Oriented Small Loop Antenna in a Cold Collisionless Magnetoplasma

THOMAS N. C. WANG AND TIMOTHY F. BELL

***Abstract*—Closed-form expressions are derived for the input impedance Z of a small loop antenna with arbitrary orientation in a cold collisionless uniform multicomponent magnetoplasma. The closed-forms results of Z are compared with numerical results from a full-wave formal solution. It is found that for small loops these approximate formulas adequately represent the loop input impedance.**

I. Introduction

The importance of studying the problem of the radiation characteristics of a loop antenna in a magnetoplasma has been made clear by a number of workers [1]–[6]. In an earlier paper [6], we investigated the characteristics of input impedance at VLF/ELF of an arbitrarily oriented loop antenna in a cold lossless uniform magnetoplasma. Using a linear electromagnetic full-wave solution, the formal integral expression for the input impedance Z of the loop antenna was developed, and a numerical integration for Z was carried out over the VLF/ELF frequency range. The purpose of this communication is to extend the work in [6] by developing approximate closed-form solutions for the loop input impedance Z in the case when the driving frequency lies between the proton gyrofrequency and the lower hybrid resonance frequency. When the driving frequency lies between the proton gyrofrequency (ω_{Hp}) and the lower hybrid resonance frequency (ω_{LHR}), the refractive index surfaces of the two characteristic electromagnetic modes are closed. In this case the integral expressions given in [6] for the radiation resistance and the reactance correction term [6, eqs. (6a) and (6b)] can be expanded in a convergent power series in increasing powers of the quantity βr, where β is the free space wavenumber and r is the loop radius. In the event that the loop radius is small enough, the first term in these series expansions will be sufficient to approximate the integrals themselves. The general condition for this approximation to hold is that:

$$(\beta r)^2[\epsilon_{+1}\sin^2\phi_0 + a\cos^2\phi_0] \ll 1 \tag{1a}$$

where ϵ_{+1}, ϕ_0, and "a" are defined in [6].

The desired approximation is obtained by making the following substitutions in [6, eqs. (6a) and (6b)]

$$J_1(V_\pm) \cong \frac{1}{2V_\pm}$$

$$\exp[-2\,|V_\pm|\sin\alpha\sin\delta] \cong 1 - 2\,|V_\pm|\sin\alpha\sin\delta$$

$$\sin[2V_\pm\sin\alpha\sin\delta] \cong 2V_\pm\sin\alpha\sin\delta \tag{1b}$$

where $V_\pm$ is proportional to the quantity βr.

II. Radiation Resistance

Given (1), [6, eq. (6a)] becomes:

$$R = R_{||}\cos^2\phi_0 + R_\perp\sin^2\phi_0 \tag{2}$$

where

$$R_{||} = C_0\int_0^{\pi/2}\frac{[\epsilon_0 - n_-^2(1 + A\cos^2\theta)]}{G(\theta)}\,n_-\sin^3\theta\,d\theta \tag{3a}$$

$$R_\perp = C_0\int_0^{\pi/2}\frac{\epsilon_0 - n_-^2(1-\frac{1}{2}\sin^2\theta) - An_-^2(\cos^2\theta + \frac{1}{2}\sin^4\theta)}{G(\theta)}\cdot n_-\sin\theta\,d\theta \tag{3b}$$

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T. N. C. Wang is with the Radio Physics Laboratory, Stanford Research Institute, Menlo Park, Calif. 94025.
T. F. Bell is with the Radio Science Laboratory, Stanford University, Stanford, Calif. 94305.

and where $A = (\epsilon_0\epsilon_s - \epsilon_{+1}\epsilon_{-1})/\epsilon_{+1}\epsilon_{-1}$, $C_0 = \pi Z_0(\beta r)^4\epsilon_{+1}\epsilon_{-1}/4$, $G(\theta) = \alpha(\theta)(n_+^2 - n_-^2)$, $\alpha(\theta) = \epsilon_0\cos^2\theta + \epsilon_s\sin^2\theta$, and $n_\pm \equiv n_\pm(\theta)$ are the refractive indices of the ordinary and extraordinary modes [7]. It is a simple matter to show that (3a) is identical to [5, eq. 8], and thus a closed form solution already exists for it. To evaluate (3b) we first express the integral in terms of a new variable $y = [n_-(\theta)]^2$. In order to do this we use the whistler mode dispersion relation [8]

$$\cos^2\theta = \frac{\epsilon_s(y-a)(y-\epsilon_0)}{(\epsilon_s-\epsilon_0)y(y-b)}$$

and the differential relation [8] which can be derived from it:

$$\frac{\sin 2\theta\,d\theta}{G(\theta)} = \frac{dy}{(\epsilon_s-\epsilon_0)(y-b)}.$$

The transformed integral has the form:

$$R_\perp = C_1\int_{\epsilon_{+1}}^{a}\frac{y^2 - c_2y + c_3}{(y-b)^{5/2}}\left[\frac{a-y}{(y-\epsilon_0)}\right]^{1/2}dy \tag{4}$$

where $y = n_-^2$, $a = \epsilon_{+1}\epsilon_{-1}/\epsilon_s$, $b = (\epsilon_0\epsilon_s - \epsilon_{+1}\epsilon_{-1})/(\epsilon_0 - \epsilon_s)$,

$$C_1 = \tfrac{3}{8}R_0\left(\frac{\epsilon_s}{\epsilon_0-\epsilon_s}\right)^{1/2}\frac{|\epsilon_0|\,[(\epsilon_0-\epsilon_s)^2+\epsilon_d^2]}{(\epsilon_s-\epsilon_0)^2},$$

$$R_0 = \frac{Z_0(\beta r)^4\pi}{6}$$

$$C_2 = \frac{2(\bar\epsilon\epsilon_s + 2\epsilon_0\epsilon_d^2)}{(\epsilon_0-\epsilon_s)^2+\epsilon_d^2}, \qquad \bar\epsilon = (\epsilon_0-\epsilon_{+1})(\epsilon_0-\epsilon_{-1})$$

$$C_3 = \frac{\bar\epsilon\epsilon_{+1}\epsilon_{-1} + 2\epsilon_0^2\epsilon_d^2}{(\epsilon_0-\epsilon_s)^2+\epsilon_d^2}, \qquad \epsilon_s = \tfrac{1}{2}(\epsilon_{+1}+\epsilon_{-1}), \qquad \epsilon_d = \tfrac{1}{2}(\epsilon_{+1}-\epsilon_{-1}).$$

For the frequency range considered the plasma parameters satisfy the inequalities: $a > \epsilon_{+1} > 0 > b \geq \epsilon_0$. Using well-known integrals involving elliptical functions, (4) can be integrated in the following closed form:

$$R_\perp = 2C_1\left\{B_1F(q,p) - B_2E(q,p) + \frac{1}{3}\left|\frac{\epsilon_{+1}}{\epsilon_s}\right|^{1/2}\cdot\left[8B_3 - \frac{\epsilon_{+1}\epsilon_d}{\epsilon_0(\epsilon_{+1}-b)}B_4\right]\right\} \tag{5}$$

where

$$B_1 = (a-\epsilon_0)^{1/2}\left[1 - \frac{\epsilon_s}{3\epsilon_0}B_4\right]$$

$$B_2 = (a-\epsilon_0)^{1/2}\left[\left(1+\frac{\epsilon_s}{3\epsilon_0}\right)B_4 + \tfrac{14}{3}B_3\right]$$

$$B_3 = \frac{\epsilon_d^2}{(\epsilon_0-\epsilon_s)^2+\epsilon_d^2}$$

$$B_4 = \frac{\bar\epsilon}{(\epsilon_0-\epsilon_s)^2+\epsilon_d^2}$$

$$p = \left(\frac{a-b}{a-\epsilon_0}\right)^{1/2}$$

and where $F(q,p)$ and $E(q,p)$ are the standard notations for the incomplete elliptical functions of the first and second kind, respectively, with argument $q = \arcsin[(a-\epsilon_{+1})/(a-b)]^{1/2}$, and p is defined in the preceding.

Equation (5) and [5, eq. (8)] can be approximated by much simpler expressions over certain frequency ranges [7]. In the interests of brevity we give these expressions here without proof.

Case A:

$$\omega \to \omega_{\mathrm{LHR}}.$$

When the driving frequency is close to the lower hybrid resonance frequency, $\epsilon_s \to 0$ and the plasma parameters have the ordering $a \gg \epsilon_{+1} > 0 > \epsilon_{-1} \gg \epsilon_0$. In this case

$$R_{||} \cong \tfrac{3}{2} R_0 \frac{\epsilon_{+1}^3}{|\epsilon_s|\,(|\epsilon_0|)^{1/2}} \cong \frac{3R_0}{2} \frac{\omega_0^3}{\omega_{He}(\omega_{LHR}^2 - \omega^2)}$$

$$R_\perp \cong \tfrac{3}{4} R_0 \,|\epsilon_0|^{1/2} \epsilon_{+1} \log\left(\frac{\epsilon_{+1}^2}{|\epsilon_s|}\right) \cong \frac{3R_0}{4} \frac{\omega_0^2}{\omega_{He}\omega^2} \log\left(\frac{\omega_0^2}{\omega_{LHR}^2 - \omega^2}\right). \quad (6)$$

Case B:

$$5\omega_{Hp} \le \omega \le \tfrac{1}{2}\omega_{LHR}.$$

When the driving frequency lies between approximately five times the proton gyrofrequency and one-half the lower hybrid resonance frequency, the following approximation holds:

$$R_{||} \cong \frac{3\pi}{4} R_0 \frac{\epsilon_{+1}^2}{(|\epsilon_s|)^{1/2}} \cong \frac{3\pi}{4} R_0 \frac{\omega_0^3}{\omega_{He}\omega_{LHR}\omega} \quad (8)$$

$$R_\perp \cong \tfrac{1}{4} R_{||}. \quad (9)$$

Case C:

$$\omega \to \omega_{Hp}.$$

When the driving frequency approaches the proton gyrofrequency from above, the following approximation holds:

$$R_{||} \cong 2R_0 \epsilon_{+1}^{3/2} \cong \frac{R_0}{\sqrt{2}} \frac{\omega_0^3}{(\omega_{He}\omega)^{3/2}}$$

$$R_\perp \cong \tfrac{1}{4} R_{||}. \quad (10)$$

In (6)–(8) and (10), ω_0 is the angular plasma frequency, ω_{He} is the angular electron gyrofrequency, and ω_{LHR} is the angular lower hybrid resonance frequency. The last expression on the right in (6)–(8) and (10) applies to a two component plasma consisting of equal numbers of electrons and protons.

It is interesting to note from (2) and (8)–(11) that the radiation resistance of the loop is not a rapidly varying function of the loop orientation angle over most of the frequency range considered here.

III. Reactance

Given (1), [6, eq. (6b)] becomes

$$X^c = \frac{4}{3\pi^2} Z_0 (\beta r)^3 \epsilon_{+1}\epsilon_{-1} \int_0^\pi d\psi \int_0^{\pi/2} d\theta \, \frac{1 + A\cos^2\theta + A\Phi}{\alpha(\theta)} \cdot (1 - \Delta^2 \sin\theta)^{1/2} \quad (12)$$

where all symbols are defined in [6]. It can be shown that (12) is identical to the quasi-static reactance correction term given in [6, eq. (5a)]. An upper bound X^B can be constructed for X^c from (12):

$$X^c \le X^B = \frac{4}{3\pi} Z_0 (\beta r)^3 (3a - 2\epsilon_0). \quad (13)$$

However, the fact that we use only the first term in the expansion shown in (1b) implies the constraint that $(\beta r)^2 a \ll 1$ and $(\beta r)^2 |\epsilon_0| \ll 1$. Thus it can be inferred from (13) that $X^c \le Z_0 \beta r$. Since the leading term of the loop reactance, as given by [6, eq. (2)], is $X^l = \beta r Z_0 \log(\beta r/\xi)$, where $r/\xi \gg 1$, it is clear that X^c is negligible compared to X^l.

IV. Discussion

The closed-form result developed in this communication apply in the frequency range between the gyrofrequency of the lightest ion and the lower hybrid resonance frequency. The following interesting features for the input impedance of a small loop antenna in a cold multicomponent magnetoplasma have been found from these results.

1) The input reactance of a small loop antenna is essentially the same as its free-space self-inductance.

2) The loop radiation resistance is not, in general, a sensitive function of the loop orientation angle, except when $\omega \sim \omega_{LHR}$.

3) In general, the small loop approximation fails as $\omega \to \omega_{LHR}$.

We compared numerical curves from the closed-form formulas to those obtained through a numerical integration of full-wave formal solutions [5] and [6] and found that they are in essential agreement. In the interests of brevity the figures for these numerical curves are omitted.

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Angle and Doppler Measurements of the Quasi-Coherent and Incoherent Components of Microwave Transhorizon Signals

NICHOLAS CIANOS AND ALAN T. WATERMAN, JR.

***Abstract*—This communication compares the differences between the quasi-coherent and incoherent components of microwave transhorizon signals where the differences are exhibited in the received-signal characteristics in several respects: the distribution of the signal level, the shape of the antenna response patterns, and the width of the RF Doppler spectrum. These differences are also accompanied by corresponding differences in the meterological profiles. There is a clear implication that the atmospheric structure giving rise to the transhorizon scatter of the propagated radio wave is markedly different for each case. The important point of this work is that the distinctions between the quasi-coherent and incoherent components of the signal are revealed in independently measured quantities.**

Introduction

This communication compares two sets of data taken from transhorizon microwave propagation measurements. The two sets of data exhibit different received-signal characteristics in several respects: distribution of signal level, shape of antenna response patterns, and width of RF Doppler spectrum. These differences are accompanied by corresponding differences in the meteorological profiles for the two cases. There is a clear implication that the atmospheric structure giving rise to the transhorizon scatter of the propagated radio wave is markedly different in the two instances. These two cases

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N. Cianos was with the Stanford Electronics Laboratories, Stanford, Calif. He is now with the Radio Physics Laboratory, Stanford Research Institute, Menlo Park, Calif. 94025.

A. T. Waterman, Jr., is with the Stanford Electronics Laboratories, Stanford, Calif. 94305.