Radiation Resistance of a Small Filamentary Loop Antenna in a Cold Multicomponent Magnetoplasma

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Abstract—A study is made of the radiation resistance R of a small filamentary loop antenna immersed in a cold collisionless uniform multicomponent magnetoplasma. Assuming that the current distribution along the loop is uniform and that the loop axis is parallel to the static magnetic field, an integral expression is derived for R which is valid for arbitrary values of driving frequency, plasma composition and density, and static magnetic field strength. The mathematical properties of this integral are such that R is finite for all values of the driving frequency, including the upper and lower hybrid frequencies as well as the multiple-ion hybrid-resonance frequencies. Application of the integral expression is made to the case of the inner magnetospheric plasma and approximate closed-form expressions are developed for R for the very low-frequency/extremely low-frequency (VLF/ELF) range in the magnetosphere. Numerical results in the VLF/ELF range are also presented.

It is found that the inclusion of multiple ions introduces interesting effects: sharp maxima appear at the gyrofrequency and the multiple-ion hybrid-resonance frequency associated with each ion, while sharp minima occur at the "crossover" frequencies. It is concluded that the presence of these relative extrema in the radiation resistance presents interesting possibilities for the use of a small loop as a diagnostic tool in a multicomponent plasma.

I. Introduction

OVER THE PAST years a considerable effort has gone into the study of the radiation characteristics of simple antennas in a magnetoplasma. The major portion of this effort has been devoted to the study of electric dipole antennas (see [1] for a comprehensive review through 1966 of papers involving plasma-immersed dipole antennas), and to date only a few studies have been devoted to magnetic loop antennas [2]–[7]. It is unfortunate from an applications point of view that not more is known of the plasma radiation characteristics of loops, since there appears to be some reason to believe that loops may be superior as radiating elements in a magnetoplasma [1].

One area in which very little work has been done and in which more study is especially needed is that involving the radiation characteristics of loops at driving frequencies less than the electron gyrofrequency in a moderate- to high-density multicomponent plasma. This study has direct application to the planning of wave-particle interaction experiments involving satellite-based very low-frequency/extremely low-frequency (VLF/ELF) transmitters in the magnetosphere. An important question to be answered in planning such experiments is: given the experimental constraints on antenna dimensions and transmitter power, for what type of antenna can the power radiated into whistler mode waves be maximized?

The answer to this question involves knowing the antenna input impedance as well as the power loss into unwanted modes, and at present only partial estimates of these quantities are available for loops at VLF/ELF in a multicomponent plasma. It is clear that much more study is needed before such experiments can be properly planned. In the present paper we hope to develop further understanding of the problem of the coupling between a VLF/ELF loop antenna and a multicomponent plasma through the consideration of an idealized case. Specifically, we calculate the radiation resistance R of a small filamentary loop of uniform current in a cold collisionless uniform multicomponent magnetoplasma.

From the results of our study it is found that the inclusion of multiple ions introduces some interesting effects. For instance, at a finite number of frequencies known as "crossover" frequencies [8], the input impedance of the loop is identical to what it would be in an isotropic medium of relative dielectric constant ϵ_{+1} (symbol defined in Section II). In the case of a small loop it is found that R has sharp local minima at these same crossover frequencies with the depth of the minima varying inversely as the loop radius. Furthermore, it is found that R has sharp local maxima at the lower hybrid-resonance frequency as well as at a finite number of other frequencies known as multiple-ion hybrid-resonance frequencies [8]. The presence of these sharp maxima and minima in R (as a function of frequency) is important to the problem of the design of a VLF/ELF satellite transmitting system and also presents interesting possibilities for the use of a small loop as a diagnostic tool in a multicomponent plasma.

II. Basic Formulation

For a circular filamentary loop of radius r oriented with its axis of symmetry parallel to the positive z axis (parallel to the static magnetic field) and carrying a uniform

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current I_0, the current density may be specified by

$$J(\varrho) = rI_0 \frac{\delta(\rho - r)\delta(z)}{\rho} \hat{\phi} \tag{1}$$

where ϱ represents the position vector of a point in cylindrical coordinates (ρ,ϕ,z) and $\hat{\phi}$ is a unit vector in the positive ϕ direction.

Considering (1) as a source function in a cold multicomponent magnetoplasma, the formulation developed by the authors in a previous paper [9] can be used to determine the mean complex radiated power from the loop

$$P = \frac{j(I_0\lambda)^2 Z_0}{4\pi} \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{n^2\alpha(\theta) - \epsilon_s\epsilon_0}{\alpha(\theta)(n^2 - n_-^2)(n^2 - n_+^2)} \cdot J_1^2(\lambda\, n \sin\theta) n^2 \sin\theta \, d\psi \, d\theta \, dn \tag{2}$$

where $n_\pm \equiv n_\pm(\theta), \epsilon_\nu = 1 - \sum[X_r/(1 + \nu Y_r)]$ (sum over species), X_r and Y_r (carrying charge sign) are standard notations for the normalized frequencies for each species in a magnetoionic medium [10], $\epsilon_s = \frac{1}{2}(\epsilon_{+1} + \epsilon_{-1})$, $\lambda = \beta r$, $Z_0 = (\mu/\epsilon)^{1/2} \cong 377\ \Omega$, and all other notation is defined in [9]. An interesting feature of (2) is the fact that whenever $\epsilon_{+1} = \epsilon_{-1}$, the mean complex power, hence the input impedance, is identical to what it would be in an isotropic medium of relative dielectric constant ϵ_{+1}. In a multicomponent plasma the condition $\epsilon_{+1} = \epsilon_{-1}$ occurs at the crossover frequencies [8], and thus this special case is of interest to the present work. The calculation of the loop radiation resistance near the crossover frequencies can be found in Section IV.

A. Full-Wave Radiation Resistance

To evaluate (2) we perform the trivial ψ integration and then perform the n integration using the technique of contour integration and assuming the medium to be slightly lossy to ensure convergence of the integral. Used in conjunction with the relation $R = 2 \operatorname{Re} P/I_0^2$ and [11, eqs. (4), (5)], the results of the contour integration lead to the following formal expression for the full-wave radiation resistance of the loop:

$$R = A\,|\epsilon_s|^{-1/2} \left[\int F(y_-) J_1^2(V_-)\, dy_- + \int F(y_+) J_1^2(V_+)\, dy_+ \right] \tag{3}$$

where

$$A = \frac{Z_0 \pi \lambda^2 \epsilon_d^2 \,|\epsilon_0|}{2\,|\epsilon_s - \epsilon_0|^{3/2}}, \qquad F(y) = \left| \frac{y - \epsilon_0}{(y - b)^3 (y - a)} \right|^{1/2}$$

$$y_\pm \equiv n_\pm^2, \qquad \epsilon_d = \tfrac{1}{2}(\epsilon_{+1} - \epsilon_{-1})$$

$$b = \frac{\epsilon_{+1}\epsilon_{-1} - \epsilon_0\epsilon_s}{\epsilon_s - \epsilon_0}, \qquad a = \frac{\epsilon_{+1}\epsilon_{-1}}{\epsilon_s}$$

$$V_\pm = \lambda \left[\frac{\epsilon_0 (y_\pm - \epsilon_{+1})(y_\pm - \epsilon_{-1})}{(\epsilon_0 - \epsilon_s)(y_\pm - b)} \right]^{1/2}$$

and the domain of $y_\pm$ for the integrals covers all positive values of $n_\pm^2$ (the region of propagation for each mode in the wave normal space).

B. Quasi-Static Radiation Resistance

The quasi-static approximation to the loop radiation resistance can be obtained from (2) by taking the leading term of the real part of P as $n \to \infty$. Assuming vanishingly small losses, this term can be easily evaluated to yield

$$R_Q = \frac{4}{3} Z_0 \frac{\epsilon_d^2 \xi^3}{[\epsilon_s(\epsilon_s - \epsilon_0)]^{1/2}}, \qquad \frac{\epsilon_s}{\epsilon_0} < 0 \tag{4}$$

where

$$\xi = \beta r \left(\frac{\epsilon_0}{\epsilon_0 - \epsilon_s} \right)^{1/2} \qquad \text{and} \qquad R_Q = 0$$

when $\epsilon_s/\epsilon_0 > 0$.

III. Boundedness of Radiation Resistance

It is not difficult to show that in general (3) is bounded for all finite values of the driving frequency ω. On the other hand, since it has been claimed by Seshadri and Tuan [7] that R is infinite at the upper hybrid-resonance frequency (ω_{UH}), it is of interest to examine here in detail the behavior of R at the hybrid frequencies.

A. Upper Hybrid Resonance

Define the frequency ω_M to be the larger of ω_{H_e} and ω_0 (electron gyrofrequency and plasma frequency, respectively). For $\omega_M < \omega < \omega_{\mathrm{UH}}$, it can be shown that n_+^2 varies from ϵ_{-1} to ϵ_0 as θ varies from 0 to $\pi/2$, and that n_-^2 varies from ∞ to a ($a \equiv \epsilon_{+1}\epsilon_{-1}/\epsilon_s$) as θ varies from θ_c (cutoff shadow boundary) to $\pi/2$.

As $\omega \to \omega_{\mathrm{UH}}$, $\epsilon_s \to 0$, $\epsilon_{+1} \to -\epsilon_{-1}$, $b \to \epsilon_{-1}^2/\epsilon_0$, $\epsilon_d^2 \to \epsilon_{-1}^2$, and $a \to \infty$. Equation (3) can then be written

$$R_{\mathrm{UH}} = \frac{A}{\epsilon_{-1}} \left[\int_{\epsilon_0}^{\epsilon_{-1}} \frac{(y_+ - \epsilon_0)^{1/2} J_1^2(V_+)}{(\epsilon_{-1}^2/\epsilon_0 - y_+)^{3/2}}\, dy_+ + \lim_{\epsilon_s = 0} \frac{\epsilon_{-1}}{|\epsilon_s|^{1/2}} \cdot \int_a^\infty \frac{(y_- - \epsilon_0)^{1/2} J_1^2(V_-)}{(y_- - a)^{1/2}(y_- - \epsilon_{-1}^2/\epsilon_0)^{3/2}}\, dy_- \right]. \tag{5}$$

From (5) it is not difficult to show that an upper bound for R at ω_{UH} is the following:

$$R_{\mathrm{UH}}^B = \frac{\pi \beta^2 r^2 Z_0 \epsilon_0}{(\epsilon_{-1})^{1/2}}. \tag{6}$$

Since $\epsilon_0/(\epsilon_{-1})^{1/2} < 1$ at ω_{UH}, R is finite at this frequency and, furthermore, may be quite small for small loops. This result is in disagreement with the findings of Seshadri and Tuan [7], who concluded on the basis of qualitative arguments that R should be infinite at ω_{UH}.

B. Lower Hybrid Resonance

The lower hybrid-resonance frequency (ω_{LH}) in a cold collisionless multicomponent magnetoplasma can be defined as the unique frequency lying between ω_{H_p} (the proton gyrofrequency) and ω_{H_e}, which satisfies the condition $\epsilon_s = 0$. The behavior of R at ω_{LH} varies according to whether the plasma frequency is higher or lower than ω_{LH}. If $\omega_0 < \omega_{\mathrm{LH}}$, it can be shown that an upper bound on R at ω_{LH} is given by (6) with ϵ_{-1} replaced by ϵ_{+1}.

On the other hand, if $\omega_0 \geq \omega_{\mathrm{LH}}$ (as will occur in plasmas of moderate to high density), it is a simple matter to show that the following expression represents an upper bound for R_{LH}:

$$R_{\mathrm{LH}}{}^B = \frac{\pi}{2} Z_0(\beta r)^2 \left[1 + \frac{8}{3\pi} \beta r \mid \epsilon_0 \mid^{1/2}\right]. \qquad (7)$$

Since $|\epsilon_0| \neq \infty$ when $\omega = \omega_{\mathrm{LH}}$, $R_{\mathrm{LH}}{}^B$ is always finite for finite values of the loop radius.

C. Multiple-Ion Hybrid Resonances

In the event that more than one positive ion is present in the plasma, an additional hybrid resonance $\epsilon_s = 0$ will appear for each additional ion. If we order the ion and electron gyrofrequencies on a frequency scale according to absolute magnitude, i.e., $|\omega_{H_e}| > |\omega_{H_1}| > |\omega_{H_2}|$, etc., it can be shown that the distribution of the hybrid-resonance frequencies is such that there is one and only one hybrid-resonance frequency located between any pair of adjacent gyrofrequencies [8]. It is possible to show that an upper bound for R near any of these multiple-ion hybrid-resonance frequencies ω_{MH_i} can be obtained from (6) if $\omega_0 < \omega_{\mathrm{MH}_i}$ (with ϵ_{-1} replaced by ϵ_{+1}) or (7) if $\omega_0 \geq \omega_{\mathrm{MH}_i}$.

On the basis of the foregoing calculations, it can be stated that a filamentary loop whose axis of symmetry is parallel to the ambient magnetic field in a cold multicomponent magnetoplasma possesses a radiation resistance which is finite for all frequencies including the upper and lower hybrid-resonance frequencies. Thus the results of Seshadri and Tuan [7], showing an infinity in R at the upper hybrid frequency, are incorrect.

IV. Applications to Magnetosphere

We wish to use (3) in order to obtain a first-order estimate of the radiation resistance of a loop antenna operating in the inner magnetosphere. We wish to do this both by developing approximate closed-form solutions for R in various regimes of the plasma parameters and also by calculating R numerically from (3).

In modeling our plasma upon the inner magnetospheric plasma, we assume that $(\omega_0/\omega_{H_e}) > \sqrt{2}$. This condition insures that only one mode, the whistler mode, can propagate for $\omega_{H_e} \geq \omega \geq \omega_{H_p}$ (proton gyrofrequency), and the condition is generally satisfied throughout the magnetosphere, so long as the observation point is located within the plasmapause [12].

A. Closed-Form Solutions

In the following we list some approximate closed-form solutions for R that can be obtained from (3) for various frequencies in the approximate VLF/ELF range $\omega_{H_e} \geq \omega \geq 0$. In the interest of brevity these solutions are given without proof.

1) *R for $\omega_{H_p} < \omega < \omega_{\mathrm{LH}}$*: For this case, given that $\beta r(a)^{1/2} \leq \frac{1}{2}$, R has the form

$$R = \frac{\pi}{12} Z_0(\beta r)^4 \left(\frac{\epsilon_d}{\epsilon_s - \epsilon_0}\right)^2 \epsilon_0 \mid b \mid^{1/2}$$
$$\cdot\left\{\left[3\frac{\epsilon_0}{\epsilon_s} + 5 + 2p^{-2}\right] E(\sigma,p) - [4 + 2p^{-2}]\right.$$
$$\left.\cdot F(\sigma,p) - \left(\frac{\epsilon_{+1}}{|b|}\right)^{1/2}\left[5 + 2p^{-2} - \left(\frac{\epsilon_0 - b}{\epsilon_{+1} - b}\right)\right]\right\} \qquad (8)$$

where

$$\sigma = \arcsin\left(\frac{a - \epsilon_{+1}}{a - b}\right)^{1/2}$$
$$p = \left(\frac{a - b}{a - \epsilon_0}\right)^{1/2}$$

and $F(\sigma,p)$ and $E(\sigma,p)$ are the standard elliptic integrals of the first kind and second kind, respectively. (If $\omega \approx \omega_{H_p}$, a simpler expression for R can be used: $R \cong (Z_0\pi/3)(\beta r)^4 \epsilon_{+1}{}^{3/2}$.)

2) *R for $\omega_{H_e} \geq \omega \geq \omega_{\mathrm{LH}}$*: For this case, given that

$$\gamma \equiv \left|\frac{\epsilon_0 - \epsilon_s}{4\epsilon_0}\right| (\beta r)^{-2} \gg 2 \mid \epsilon_d \mid \qquad (9)$$

the following approximate closed-form expressions for R can be obtained.

Case I) $\gamma \gg \max(|\epsilon_0|, |a|)$: For any given finite values of ϵ_0 and a, the loop can be made small enough so that the condition $\gamma \gg \max(|\epsilon_0|, |a|)$ will obtain. In this limit of small loop dimensions R_1 becomes identical to the quasi-static result (4).

Case II) $\gamma \ll \min(|a|, |\epsilon_0|)$: In the event that the loop is large enough so that $\gamma \ll \min(|a|, |\epsilon_0|)$, the following expression is obtained for R:

$$R_2 \cong \frac{4}{3} Z_0 \frac{\epsilon_d{}^2\xi^3}{(\epsilon_s - \epsilon_0)^{1/2}} \left(\frac{\epsilon_0}{\epsilon_{+1}\epsilon_{-1}}\right)^{1/2}. \qquad (10)$$

Case III) $|a| \gg \gamma \gg |\epsilon_0|$: If a loop of small but finite dimensions (such that $\gamma \gg |\epsilon_0|$) is driven at a frequency approaching ω_{LH}, then since $|a| \rightarrow \infty$ as $\omega \rightarrow \omega_{\mathrm{LH}}$, the condition must eventually obtain: $|a| \gg \gamma \gg |\epsilon_0|$. In this case R has the form

$$R_3(\omega \sim \omega_{\mathrm{LH}}) \approx \frac{\pi Z_0}{2} \frac{(\beta r \epsilon_d)^2}{(\epsilon_{+1}\epsilon_{-1}\epsilon_0)^{1/2}}. \qquad (11)$$

Since R_3 varies as $(\beta r)^2$ while R_1 (Case I) varies as $(\beta r)^3$, $R(\omega = \omega_{\mathrm{LH}})$ can be arbitrarily large compared with $R(\omega \neq \omega_{\mathrm{LH}})$ as $r \rightarrow 0$. Consequently, R can exhibit a relative maximum when $\omega = \omega_{\mathrm{LH}}$. It can be shown that (11) is valid only if $\lambda^2 |a| \gg 1$. Thus the relative maximum at $\omega = \omega_{\mathrm{LHR}}$ will be very narrow.

Case IV) $a \cong \epsilon_0$: In the inner magnetosphere a reasonable approximation over a significant portion of the VLF range is the assumption $a \cong \epsilon_0$. Given this assumption, R is found to be identical to (4). Note that the relation $a \cong \epsilon_0$ does not directly imply any constraint on the loop radius.

3) *R for $\omega_{H_p} \geq \omega > 0$*: It is convenient to speak in terms of a set of "basic" solutions for R which can be applied in one combination or another to cover the entire frequency range $\omega_{H_p} \geq \omega \geq 0$. The necessary set can be deduced by consideration of the ordering of the plasma parameters ϵ_{+1}, ϵ_{-1}, ϵ_s and ϵ_0 as ω is decreased monotonically from a given ion gyrofrequency ω_{H_i} to the next lowest ion gyrofrequency $\omega_{H_{i+1}}$. If we assume a plasma of moderate to high density so that $\epsilon_0 < 0$, for $\omega \leq \omega_{H_p}$, and assume the absence of negative ions so that $\epsilon_{+1} > 0$, it is found that only the following orderings need be considered.

a) $\epsilon_{-1} \geq \epsilon_s \geq \epsilon_{+1} > 0$: The frequency range associated with this regime is $\omega_{H_i} > \omega > \Omega_{c_i}$, where Ω_{c_i} is the crossover frequency [8]. The crossover frequencies are defined as the set of real positive frequencies which satisfy the relation $\epsilon_{+1} = \epsilon_{-1}$. There is one and only one crossover frequency between ω_{H_i} and $\omega_{H_{i+1}}$, and at this frequency the $n_\pm$ modes exchange identity (based on polarization labeling).

The n_- mode contribution to R for this regime has the form

$$R_- = \frac{Z_0 \pi \lambda^4 \epsilon_{+1}^{3/2}}{6(\epsilon_s - \epsilon_0)} \left[\epsilon_{-1} - \epsilon_0 - \frac{3}{2} \frac{\epsilon_d \epsilon_0^2}{\epsilon_s(\epsilon_s - \epsilon_0)} \right]. \quad (12)$$

In the case of a small loop the leading term of R due to the n_+ mode can be obtained from (4). Thus to a good approximation the radiation resistance R_a of a small loop in this frequency range is given by the sum of (4) and (12). Note that as $\omega \to \Omega_{c_i}$, $\epsilon_{+1} \to \epsilon_{-1}$, $\epsilon_d \to 0$, and R_a assumes the value appropriate to an isotropic medium of relative dielectric constant ϵ_{+1}, i.e., $R_a(\omega = \Omega_{c_i}) = (Z_0\pi/6)(\beta r)^4 \epsilon_{+1}^{3/2}$.

b) $\epsilon_{+1} > \epsilon_s > \epsilon_{-1} > 0$: The frequency range associated with this regime is $\Omega_{c_i} > \omega > \omega_i^c$, where ω_i^c is the n_+ mode cutoff frequency. The cutoff frequencies are defined as the set of real positive frequencies which satisfy the relation $\epsilon_{-1} = 0$. There is one and only one cutoff frequency between each two adjacent ion gyrofrequencies ω_{H_i} and $\omega_{H_{i+1}}$. Since the $n_\pm$ modes exchange identity as the driving frequency passes through the crossover frequency, R_b is given by the sum of (4) and (12) with ϵ_{+1} and ϵ_{-1} interchanged.

c) $\epsilon_{+1} \geq \epsilon_s \geq 0 \geq \epsilon_{-1}$: The frequency range of this regime is $\omega_i^c \geq \omega \geq \omega_{MH_i}$, and the radiation resistance R_c is given by either (4), (10) or (11).

d) $\epsilon_{+1} > 0 > \epsilon_s > \epsilon_{-1}$: The frequency range of this regime is $\omega_{MH_i} > \omega \geq \omega_{H_{i+1}}$, and the radiation resistance R_d is identical to that given in (8).

The preceding ordering sequence will repeat for each pair of adjacent ion gyrofrequencies until ω decreases below the smallest ion gyrofrequency ω_{H_s}. When $\omega < \omega_{H_s}$ the sequence starts once again with the ordering $\epsilon_{-1} > \epsilon_s > \epsilon_{+1} > 0$, but this ordering does not change over the range $0 \leq \omega < \omega_{H_s}$ since no crossover frequency exists in this range. Thus these orderings cover all possibilities assuming no negative ions ($\epsilon_{+1} > 0$) and moderate- to high-plasma density ($\epsilon_0 < 0$; $\omega \leq \omega_{H_p}$).

B. Numerical Results

Our numerical data have been obtained by evaluating (3) through the use of computer integration techniques. In Figs. 1–3 we plot R as a function of frequency for various values of plasma density, plasma composition, static magnetic field strength, and loop radius. The curves in each figure are parametric in the variables r_0 and f_0/f_{H_e}, where $r_0 \equiv 2\pi f_{H_e} r/c$, f_{H_e} is the electron gyrofrequency, and f_0 is the plasma frequency. The three values of f_0/f_{H_e} that are used ($f_0/f_{H_e} = 2, 5, 10$) were chosen as being representative of the range of values that can be encountered in the inner magnetosphere.

Fig. 1 is a plot of R versus normalized frequency over the range $1 \geq f/f_{H_e} \geq 2.5 \times 10^{-2}$. The plasma is assumed to consist of electrons and protons. The lowest frequency plotted is sufficiently above the lower hybrid-resonance frequency (appropriate to each value of f_0/f_{H_e}) so that ion effects are not dramatic. For each curve it can be seen that R is an increasing function of frequency until $f \sim 0.8 f_{H_e}$, at which point R reaches a maximum. For larger values of frequency R is a decreasing function of frequency which approaches zero rapidly as $f \to f_{H_e}$. When $f_0/f_{H_e} \leq 10$ and $r_0 \leq 0.1$, the curves agree with the approximate formula (4) within a few percent. In this range of the parameters R as a function of the loop radius varies as r^3 and as a function of density varies approximately as $(f_0/f_{H_e})^2$.

The curves of Fig. 1 are plotted under the assumption that the loop possesses a uniform current distribution. This assumption appears reasonable at any given frequency so long as the loop is small compared to the maximum wavelength of the medium at that frequency. However, for values of r_0 greater than 1 it can be shown that even at the lowest density ($f_0/f_{H_e} = 2$), the loop radius would exceed the maximum wavelength in the medium over most of the frequency range of Fig. 1. Consequently, values of r_0 greater than 1 were not considered.

Fig. 2 is a plot of R versus normalized frequency over the range $2 \times 10^{-3} \leq f/f_{H_e} \leq 3 \times 10^{-2}$. The plasma is assumed to consist of electrons and protons. The frequency range of this plot includes the three lower hybrid-resonance frequencies ($f_{LH}/f_{H_e} \cong 2.07 \times 10^{-2}$, 2.28×10^{-2}, 2.33×10^{-2}) appropriate to the three normalized densities ($f_0/f_{H_e} = 2, 5, 10$), and ion effects at these frequencies are noticeable on all curves plotted. The resonance behavior of R at $f = f_{LH}$ is particularly dramatic at small values of r_0, where the resonance peak is orders of magnitude higher than values of R to either side of the peak.

An interesting feature of the curve family is the tendency for the LH resonance peak to wash out as either the loop radius or the electron density is increased. An explanation of this effect can be obtained by comparing

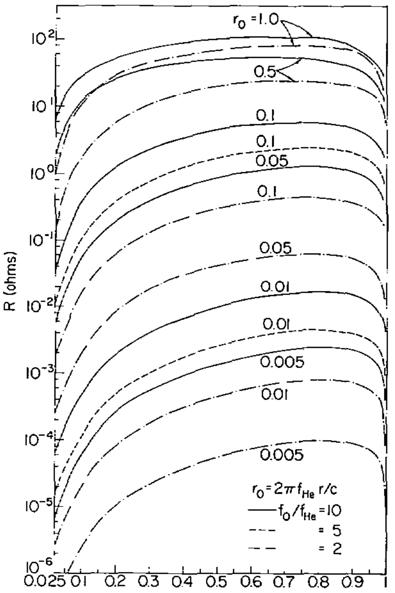

Fig. 1. Loop radiation resistance as function of frequency for various values of normalized radius r_0 and density ratio f_0/f_{He}. Two-specie plasma is assumed: electrons and protons.

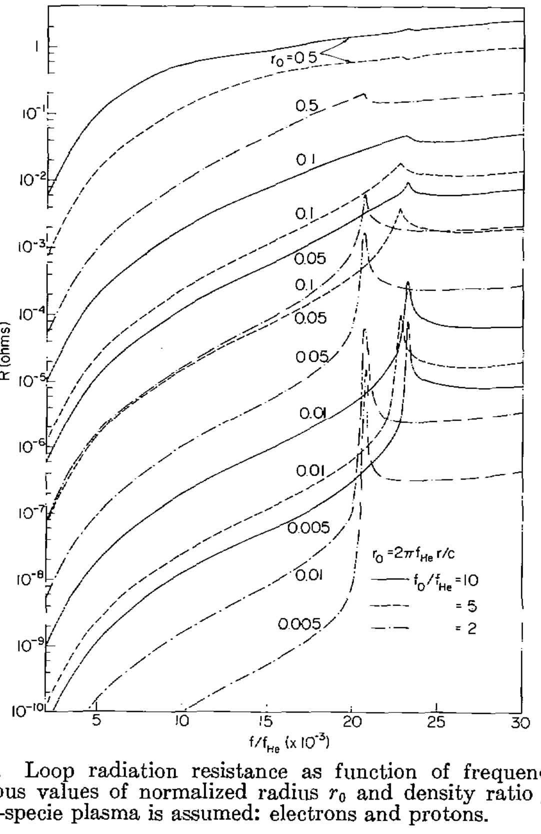

Fig. 2. Loop radiation resistance as function of frequency for various values of normalized radius r_0 and density ratio f_0/f_{He}. Two-specie plasma is assumed: electrons and protons.

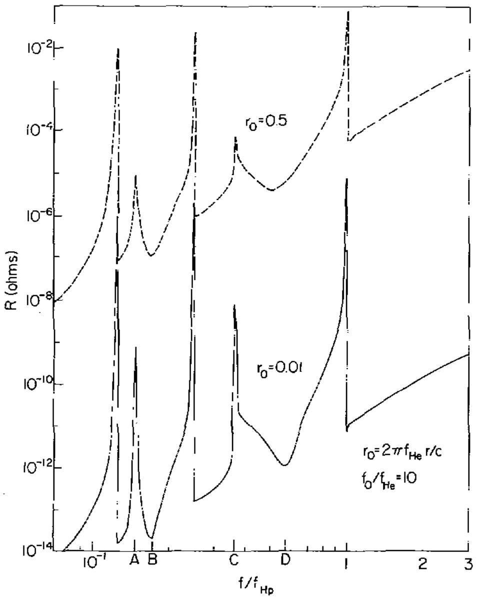

Fig. 3. Loop radiation resistance as function of frequency in multi-ion plasma. Four-specie plasma is assumed: electrons, protons (70 percent), He^+ (20 percent), and O^{++} (10 percent).

the approximate solutions (4) and (11), both of which will be quite accurate for $r_0 \leq 0.05$. Equation (4) can be used to evaluate R at the relative minimum which occurs on the high-frequency side of f_{LH}, while (11) can be used to determine the peak value. It is found that the peak value varies approximately as $r^2 f_0/f_{H_e}$, while the minimum value varies approximately as $r^3(f_0/f_{H_e})^2$; thus the relative height of the peak varies as $(rf_0)^{-1}$, and the peak tends to disappear as either the radius or density is increased.

For $\omega < \omega_{LH}$, R decreases monotonically as the frequency decreases. In this range a comparison of the approximate solution for R given in (8) with the plotted curves shows that (8) is quite accurate over the range $2 \times 10^{-3} f_{H_e} \leq f \leq 0.98 f_{LH}$ as long as $r_0 f_0/f_{H_e} \leq 0.1$.

Fig. 3 shows a plot of R versus normalized frequency over the range $7 \times 10^{-2} \leq f/f_{H_p} \leq 1$, where f_{H_p} is the proton gyrofrequency. Two curves are plotted, one for $r_0 = 0.5$ and one for $r_0 = 0.01$; in each case it is assumed that $f_0/f_{H_e} = 10$. The plasma is assumed to consist of electrons and three ions: atomic hydrogen, atomic helium, and atomic oxygen. The composition of the plasma is assumed to be 70-percent H^+, 20-percent He^+, and 10-percent O^{++}. The hydrogen and helium percentages are in line with those that can be found in the topside ionosphere (~1000 km) at night. The doubly ionized atomic oxygen is included for numerical convenience and not on the basis of any model of the ionosphere. The effects on R due to the presence of multiple ions are clearly visible in both curves of Fig. 3, but particularly so in the curve for the smaller radius $r_0 = 0.01$.

Both curves show large resonance peaks occurring at the multiple-ion hybrid-resonance frequencies (points labeled A and C on the frequency axis) and both curves show minima near the crossover frequencies (points labeled B and D on the frequency axis). However, the resonance peaks are higher and the crossover minima are deeper and sharper at the smaller radius. In addition the crossover minima for the larger radius plot show a significant shift away from the true crossover frequency location. Numerical data not shown here indicate that the smoothing of the relative extrema continues to occur as r_0 is increased.

It can be seen from the plots that relative maxima in R also occur at frequencies very close to, but slightly below, each ion gyrofrequency. Thus between each pair of adjacent ion gyrofrequencies two sharp relative maxima and one sharp relative minima are located, and in this sense the curves of Fig. 3 may be taken as representative of the behavior of R for small loops in a multicomponent plasma.

V. Discussion

To the authors' knowledge the present paper is the first published work which uses a full-wave formulation to treat the subject of the radiation resistance of a finite loop in a multicomponent plasma. Previous authors who have studied the loop problem have done so using considerably simplified models [2]–[5], [7], [13] which in the main fail to describe the interesting radiation effects produced by the ions in the VLF/ELF frequency range.

In a recent paper in the field, Duff and Mittra [6] calculated the input impedance of a small loop in a cold magnetoplasma using both a uniaxial approximation and an approximate quasi-static theory. Both approximations lead to closed-form expressions for R for the parallel loop, but a comparison with our own calculations shows their expressions to be significantly in error over most of the VLF/ELF range. As a result of these discrepancies, we infer that neither approximation in [6] is sufficiently accurate to allow a meaningful evaluation of the loop radiation resistance.

The usefulness of the closed-form expressions for R which were developed in Section IV-A depends upon the possibility of satisfying the restriction $\beta r(a)^{1/2} \leq \frac{1}{2}$ or that given in (9). Similar restrictions have been discussed in earlier papers [9], [11], and by applying the results of these papers to the present problem it is possible to show that our expressions for R will be valid for loop dimensions of the order of, or greater than, those presently used in VLF/ELF magnetospheric satellite experiments.

The presence in R of sharp relative maxima at the ion gyrofrequencies and hybrid-resonance frequencies and sharp relative minima at the crossover frequencies is clearly of importance to the problem of designing a VLF/ELF satellite transmitting system in the magnetosphere and also presents interesting possibilities for the use of a small loop as a diagnostic tool in a multicomponent plasma. One scheme that might be practical would involve using a small loop as a sweep-frequency radiating element immersed in the plasma. The far-field signal would be detected by two equidistant receivers: one positioned approximately on the same magnetic field line as the loop, and the other positioned in a direction inclined approximately 50–90° away from the field line. (Two receivers are necessary since the radiation from the loop flows almost directly along the magnetic field lines when $\omega \approx \omega_{MH_i}$, while it flows at a steadily increasing inclination to the field as $\omega \rightarrow \omega_{H_i}$.) Assuming that ω_0 and ω_{H_e} are known, the location of the ion gyro, hybrid-resonance, and crossover extrema then determines the ion masses and densities in the intervening plasma (in some average sense). An experiment of this sort might be conveniently carried out from one of the manned space platforms presently proposed by NASA, using two of the proposed remote-maneuvering vehicles to receive the far-field signals from the loop. Needless to say, this scheme is just one of a host of interesting plasma diagnostic experiments which could be performed using a VLF transmitter on a manned space platform.

Although we have neglected complicating effects due to collisions, finite temperature, inhomogeneities, and nonlinearities in the plasma, the results presented here should prove useful in providing a first-order look at the problem of the coupling between a loop antenna and a multicomponent magnetoplasma.

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