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On VLF radiation resistance of an electric dipole in a cold magnetoplasma

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By using full-wave theory, an analysis is made of the radiation resistance of a short filamentary electric dipole, oriented with an arbitrary angle with respect to the static magnetic field, in a cold, uniform magnetoplasma. The frequency range considered lies below the local lower hybrid resonance frequency and above the proton gyrofrequency, and in this range approximate closed-form expressions for the radiation resistance are obtained by using a plasma model appropriate to the magnetosphere. These closed-form expressions are valid for dipoles of moderately restricted length, and the physical implications of this length restriction are discussed. It is found that the radiation resistance R increases rapidly as ϕ_0, the angle of dipole orientation with respect to the magnetic field, varies from 0° to 30° but only gradually as ϕ_0 varies from 30° up to 90°. The ratio of $R(\phi_0 = 90°)/R(\phi_0 = 0°)$ is approximately equal to $(f_{He}/f)^2$. Except for the case of parallel orientation and $f < 10^{-2} f_{He}$, the radiation resistance of an electric dipole in the magnetoplasma is $\sim 10^2$ to 10^5 times larger than that in free space. Thus, for the low frequency range considered, an electric dipole in the magnetoplasma is generally much more efficient that it would be in free space.

INTRODUCTION

In an earlier paper [*Wang and Bell,* 1969], the authors have used a full-wave treatment to investigate the small-signal VLF radiation resistance of an electric dipole immersed in a uniform, cold magnetoplasma modeled on the inner magnetosphere. Approximate closed-form expressions were derived for the radiation resistance of an electric dipole antenna oriented either parallel or perpendicular to the static magnetic field for VLF frequencies that lie well above the lower hybrid resonance frequency. The results from the full-wave analysis led us to conclude that, up to 'moderate' antenna length, the radiation resistance can be adequately predicted by the quasi-static theory, and the correction to the quasi-static radiation resistance due to the wave fields is in general small. It is clear that a similar conclusion cannot be made for VLF frequencies below the lower hybrid resonance frequency, since in this range the input impedance calculated from a quasi-static approximation [*Balmain,* 1964] is a pure imaginary number and the resistive part is zero. The first-order radiation resistance must then be calculated from the wave fields. It is the purpose of the present paper to extend the full-wave analysis of the previous paper (*Wang and Bell,* [1969], hereafter denoted by 'paper I') and to derive closed-form expressions of the radiation resistance for frequencies below f_{LHR} (LHR is the lower hybrid resonance frequency).

Although complicating effects due to finite temperature and nonlinearities in the plasma are not treated here, our closed-form expressions should prove useful in providing 'first-order' insight into the problem of the coupling between a satellite VLF antenna system and the magnetospheric plasma.

FULL-WAVE ANALYSIS

In paper I the formal integral representation of the radiation resistance of an electric dipole antenna with a triangular current distribution was derived for both parallel and perpendicular orientation. In this section, the analysis is devoted to the derivation of approximate closed-form expressions for the radiation resistance of a similar antenna for frequencies that lie below f_{LHR} but above the proton gyrofrequency.

Parallel orientation

For the case of a filamentary dipole antenna oriented parallel to the static magnetic field, the formal solution of $R_\parallel$ can be written (see equation 23 in paper I)

$$R_\parallel = \frac{3}{2} R_0 \int_0^{\pi/2} \frac{n^3(n^2 - \epsilon_{+1})(n^2 - \epsilon_{-1})}{G(\theta)(n^2 - \epsilon_0)} \cdot \left(\frac{\sin \lambda n_\parallel}{\lambda n_\parallel}\right)^4 \cos^2\theta \sin\theta\, d\theta \qquad (1)$$

where $R_0 = [Z_0\, (h\beta)^2]/6\pi$, $n = n(\theta)$, $n_\parallel = n \cos\theta$, $\lambda = h\beta/2$, h is the antenna half-length, and the other notations are identical to those defined in paper I. As discussed in paper I, (1) gives the full radiation resistance as long as: $f_{Hp} \le f \le f_{He}$ and $f_0^2 \gg f_{He}^2$, where f_{He} and f_{Hp} are the electron and proton gyrofrequencies used, and f_0 is the plasma frequency.

It can be shown that within the frequency range of consideration ($f_{Hp} < f < f_{\rm LHR}$), n^2 varies from ϵ_{+1} to $\epsilon_{+1}\epsilon_{-1}/\epsilon_s$ and $n_\parallel$ from $(\epsilon_{+1})^{1/2}$ to 0 as θ varies from 0 to $\pi/2$. It is clear that if we set a constraint on the antenna length such that

$$h^2\beta^2\epsilon_{+1} \ll 1 \qquad (2)$$

the sine function in (1) can then be closely approximated by its argument. This approximation leads to the following form:

$$R_\parallel = \frac{3}{2} R_0 \int_0^{\pi/2} \frac{n^3(n^2 - \epsilon_{+1})(n^2 - \epsilon_{-1})}{(n^2 - \epsilon_0)G(\theta)} \cos^2\theta \sin\theta\, d\theta \qquad (3)$$

Equation 3 can be expressed in terms of a new variable, y ($\equiv n^2$), by making use of the dispersion relation (see expression 11 in paper I) for the refractive index

$$\cos^2\theta = \frac{\epsilon_s}{\epsilon_s - \epsilon_0} \frac{(y - a)(y - \epsilon_0)}{y(y - b)} \qquad (4)$$

as well as the differential relation that can be derived from (4)

$$\frac{\sin 2\theta\, d\theta}{G(\theta)} = \frac{dy}{(\epsilon_s - \epsilon_0)y(y - b)} \qquad (5)$$

where

$$a = \epsilon_{+1}\epsilon_{-1}/\epsilon_s, \quad b = (\epsilon_{+1}\epsilon_{-1} - \epsilon_0\epsilon_s)/(\epsilon_s - \epsilon_0),$$
$$\text{and} \quad y \equiv n^2$$

By using (4) and (5), (3) becomes

$$R_\parallel = \frac{3}{4} R_0 \left(\frac{\epsilon_s}{(\epsilon_0 - \epsilon_s)^3}\right)^{1/2} \cdot \int_{\epsilon_{+1}}^{a} \frac{(a - y)(y - \epsilon_{+1})(y - \epsilon_{-1})}{[(a - y)(y - b)^3(y - \epsilon_0)]^{1/2}}\, dy \qquad (6)$$

For the frequency range considered, we have $a > \epsilon_{+1} > 0 > b \ge \epsilon_0$, and by using tabulated integrals involving elliptical functions [*Gradshteyn and Ryzhik*, 1965], (6) can be integrated in the following closed form:

$$R_\parallel = R_0 \left[\frac{\epsilon_s(a - \epsilon_0)}{(\epsilon_0 - \epsilon_s)^3}\right]^{1/2} \{ AE(q, p) + BF(q, p) + C[(a - \epsilon_{+1})(\epsilon_{+1} - b)(\epsilon_{+1} - \epsilon_0)]^{1/2} \} \qquad (7)$$

where

$$A = \frac{3}{4}\left(\frac{\epsilon_{+1}\epsilon_{-1} - 4b\epsilon_s + 2b^2 + 2\epsilon_0\epsilon_s - \epsilon_0^2}{(\epsilon_0 - b)} + \frac{a + \epsilon_0 - 2b}{3}\right)$$

$$B = \frac{\epsilon_0 - 3\epsilon_s + 2b}{2}$$

$$C = \frac{3\epsilon_{+1}\epsilon_{-1} - 6\epsilon_s b + 4b^2 + \epsilon_{+1}(\epsilon_0 - b) - \epsilon_0 b}{4(b - \epsilon_0)(\epsilon_{+1} - b)[(a - \epsilon_0)]^{1/2}}$$

$F(q, p)$ and $E(q, p)$ are the general elliptical integrals of the first and second kind, respectively, with the arguments q, p defined by

$$q = \arcsin\left(\frac{a - \epsilon_{+1}}{a - b}\right)^{1/2}, \qquad p = \left(\frac{a - b}{a - \epsilon_0}\right)^{1/2}$$

The solution given in (7) is appropriate as long as $b \ge \epsilon_0$, and this condition is met for all frequencies in the range $f_{\rm LHR} \ge f \ge f_{Hp}(1 + \epsilon)$, where $\epsilon \sim 0(10^{-3})$.

Perpendicular orientation

For this case, the antenna is oriented along the positive x axis, and the static magnetic field is along the z axis. The formal solution for the radiation resistance of a filamentary dipole is written

$$R_\perp = \frac{3R_0}{4\pi} \int_0^{2\pi} d\psi \int_0^{\pi/2} d\theta \frac{n^3(n^2 - \epsilon_0)\sin^3\theta}{G(\theta)} \cdot \left[\cos^2\psi + \frac{\epsilon_d^2}{(n^2 - \epsilon_{+1})(n^2 - \epsilon_{-1})}\right] \frac{\sin^4 \lambda n_x}{(\lambda n_x)^4} \qquad (8)$$

where $n_x = n(\theta) \sin\theta \cos\psi$, and the remaining notation is defined in paper I.

Equation 8 gives the total $R_\perp$ as long as $f_0^2 \gg f_H^2$, and $f_{Hp} \le f \le f_{He}$. It can be shown that both $n(\theta)$ and n_x are bounded by 'a' as θ varies from 0 to $\pi/2$, and thus the sine function in (8) can be closely approximated by its argument whenever the following condition holds:

$$h^2\beta^2 a \ll 1 \qquad (9)$$

With the constraint of (9), the ψ integration can be easily performed, and the leading term of (8) becomes

$$R_\perp = \frac{3}{2} R_0 \int_0^{\pi/2} \frac{n^3(n^2 - \epsilon_0)\sin^3\theta}{G(\theta)} \cdot \left[\frac{1}{2} + \frac{\epsilon_d^2}{(n^2 - \epsilon_{+1})(n^2 - \epsilon_{-1})}\right] d\theta \quad (10)$$

By using (4) and (5), (10) can be written

$$R_\perp = \frac{3}{4} R_0 \frac{|\epsilon_0|}{[\epsilon_s(\epsilon_0 - \epsilon_s)^3]^{1/2}} \cdot \left\{\int_{\epsilon_{+1}}^{a} \frac{(y - \epsilon_0)(y - \epsilon_{+1})(y - \epsilon_{-1})}{[(y - b)^3(y - \epsilon_0)(a - y)]^{1/2}} dy + 2\epsilon_d^2 \int_{\epsilon_{+1}}^{a} \left[\frac{y - \epsilon_0}{(y - b)^3(a - y)}\right]^{1/2} dy\right\} \quad (11)$$

Equation 11 can be integrated to give

$$R_\perp = R_0 \frac{|\epsilon_0| (a - \epsilon_0)^{1/2}}{[\epsilon_s(\epsilon_0 - \epsilon_s)^3]^{1/2}} \{A'E(q, p) + B'F(q, p) + C'[(a - \epsilon_{+1})(\epsilon_{+1} - b)(\epsilon_{+1} - \epsilon_0)]^{1/2}\} \quad (12)$$

where

$$A' = \frac{3}{4}\left(2(b - \epsilon_s) + \frac{2a - \epsilon_0 - b}{3} - \frac{\epsilon_{+1}\epsilon_{-1} + 2\epsilon_d^2 - 2\epsilon_s b + b^2}{a - b}\right)$$

$$B' = \frac{3}{4}\left(\frac{2\epsilon_d^2 + \epsilon_{+1}\epsilon_{-1} - 2\epsilon_s b + b^2}{a - b} + \frac{\epsilon_0 - b}{3}\right)$$

$$C' = \frac{3}{4(a - \epsilon_0)^{1/2}}\left[\frac{2\epsilon_d^2 + \epsilon_{+1}\epsilon_{-1} - 2\epsilon_s b + b^2}{(a - b)(\epsilon_{+1} - b)} + \frac{1}{3}\right]$$

and $E(q, p)$, $F(q, p)$ are the elliptic integrals defined previously in relation to (7).

The solution given in (12) is appropriate as long as $b \geq \epsilon_0$ and thus applies to frequencies over the same range as described following (7). It should be noted, however, that (12) will not be correct for a dipole of finite dimensions as $f \to f_{LHR}$, since $a \to \infty$ as $f \to f_{LHR}$ and (9) cannot be satisfied. On the other hand, (12) is still useful as $f \to f_{LHR}$, since it serves as an upper bound to (8).

Arbitrary orientation

For a filamentary dipole oriented at an arbitrary angle ϕ_0 with respect to the static magnetic field, the basic formulation developed in paper I can be generalized to yield the following formal integral representation of the radiation resistance:

$$R = (3R_0/\pi)[I(\phi_0) + I(-\phi_0)] \quad (13)$$

where

$$I(\phi_0) = \cos^2\phi_0 \int_0^{\pi/2} d\psi \cdot \int_0^{\pi/2} \frac{n^3(n^2 - \epsilon_{+1})(n^2 - \epsilon_{-1})}{(n^2 - \epsilon_0)G(\theta)} S(\theta, \psi, \phi_0) \cdot \cos^2\theta \sin\theta\, d\theta$$
$$+ \sin^2\phi_0 \int_0^{\pi/2} d\psi \cdot \int_0^{\pi/2} \frac{n^3(n^2 - \epsilon_0)\sin^3\theta}{G(\theta)} \left[\cos^2\psi + \frac{\epsilon_d^2}{(n^2 - \epsilon_{+1})(n^2 - \epsilon_{-1})}\right] S(\theta, \psi, \phi_0)\, d\theta$$
$$+ \sin 2\phi_0 \int_0^{\pi/2} \cos\psi\, d\psi \cdot \int_0^{\pi/2} \frac{n^3(n^2 - \epsilon_s)\sin 2\theta \sin\theta}{2G(\theta)} S(\theta, \psi, \phi_0)\, d\theta \quad (14a)$$

and

$$S(\theta, \psi, \phi_0) = \left[\frac{\sin(\lambda\, \delta n)}{\lambda \delta n}\right]^4 \quad (14b)$$

for $\delta = \sin\theta \cos\psi \sin\phi_0 + \cos\theta \cos\phi_0$

Integration of (14), in general, is difficult. However, for an antenna length subject to the following constraint:

$$h^2\beta^2(\epsilon_{+1}\cos^2\phi_0 + a\sin^2\phi_0) \ll 1 \quad (15)$$

the small-argument expansion of $S(\theta, \psi, \phi_0)$ can be used. The leading term from the integration of (14) then yields

$$R = \cos^2\phi_0 R_\parallel + \sin^2\phi_0 R_\perp \quad (16)$$

where $R_\parallel$ and $R_\perp$ are defined as in (7) and (12).

Special cases

Intermediate frequencies. Equations 7 and 12 can be closely approximated by relatively simple expressions over the frequency range

$$5f_{H_p} \leq f \leq \tfrac{1}{2}f_{LHR} \quad (17)$$

Over this range, $a^2 \ll \epsilon_0^2$ and the factor $(y - \epsilon_0)^{1/2}$ in (7) and (12) can be well approximated by the constant $|\epsilon_0|^{1/2}$. This substitution leads to the following simple expressions for the leading terms of (7) and (12):

$$R_{\parallel} \cong \frac{3\pi}{32} R_0 \frac{\epsilon_{+1}^4}{\epsilon_0^2 |\epsilon_s|^{3/2}}$$
$$R_{\perp} \cong \frac{3\pi}{8} R_0 \frac{\epsilon_{+1}^2}{|\epsilon_s|^{3/2}} \tag{18}$$

where we have used the fact that over the frequency range of (17), $\epsilon_{+1} \cong -\epsilon_{-1}$, ${\epsilon_{+1}}^2 \gg {\epsilon_s}^2$, and $|\epsilon_d| \cong \epsilon_{+1}$.

For a plasma in which protons are the dominant ion, (18) also has a simple form in terms of basic plasma parameters

$$R_{\parallel} \cong \frac{3\pi}{32} R_0 f_0/f_{He}(f/f_{\mathrm{LHR}})^3$$
$$R_{\perp} \cong \frac{3\pi}{8} R_0 \frac{f_0 f_{He} f}{(f_{\mathrm{LHR}})^3} \tag{19}$$

where f_0 is the plasma frequency, $f^2_{\mathrm{LHR}} = f_{He} f_{Hp}$, and $(f_0/f_{He})^2 \gg 1$ is assumed.

Frequencies close to the proton gyrofrequency. For these frequencies the parameters ϵ_{-1}, ϵ_0, ϵ_s and 'b' are all large compared with 'a' and in this case (7) and (12) yield

$$R_{\parallel} \cong \tfrac{2}{5} R_0 \epsilon_{+1}^{5/2}/\epsilon_0^2$$
$$R_{\perp} \cong 3R_0 \epsilon_{+1}^{1/2} \tag{20}$$

In terms of basic plasma parameters,

$$R_{\parallel} \cong \frac{2R_0}{5} \frac{f_0 f^{3/2}}{f_{He}^{5/2}}$$
$$R_{\perp} \cong 3R_0 \frac{f_0}{(f f_{He})^{1/2}} \tag{21}$$

Frequencies close to the lower hybrid resonance frequency. For these frequencies, $\epsilon_{+1} \to -\epsilon_{-1}$, $\epsilon_s \to 0$, a $\to \infty$, and the leading terms of (7) and (12) become

$$R_{\parallel} \cong \tfrac{1}{2} R_0 \frac{\epsilon_{+1}^3}{|\epsilon_0|^{3/2} \epsilon_s} \cong \tfrac{1}{2} R_0 \frac{f_0 f^2}{f_{He}(f_{\mathrm{LHR}}^2 - f^2)} \tag{22a}$$

$$R_{\perp} \cong R_0 \frac{\epsilon_{+1}^3}{|\epsilon_0|^{1/2} \epsilon_s^2} \cong R_0 \frac{f_0 f_{He} f^2}{(f_{\mathrm{LHR}}^2 - f^2)^2} \tag{22b}$$

where $(f_0/f_{He})^2 \gg 1$.

NUMERICAL RESULTS AND DISCUSSION

To construct numerical plots, we normalize (7), (12), and (16) by dividing both sides of each equation by the radiation resistance of a short dipole in free space (i.e., $R_o = (h\beta)^2 Z_0/6\pi$). In Figures 1 and 2

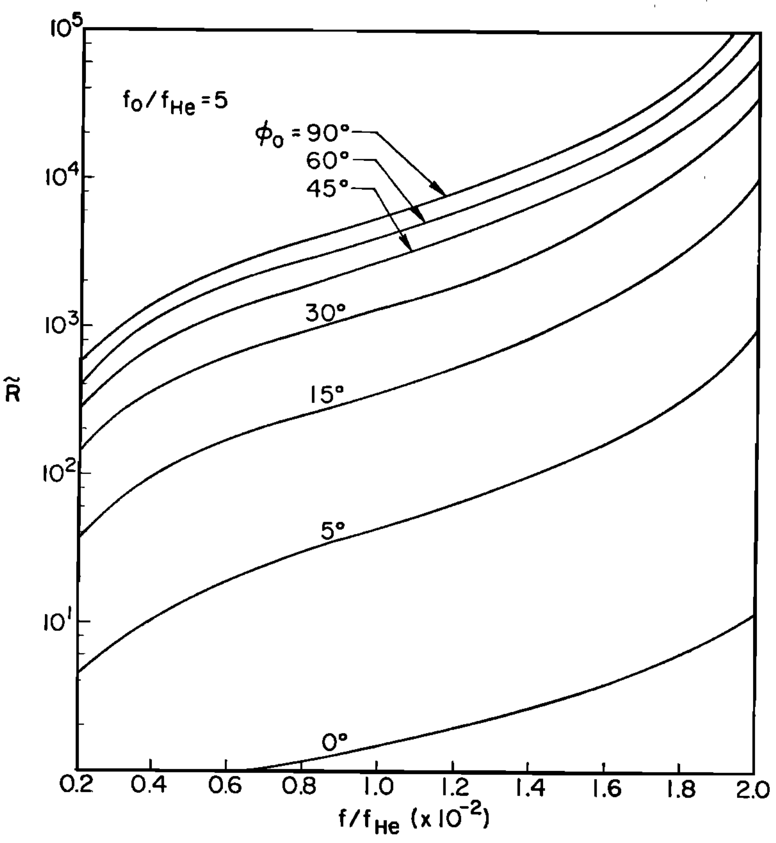

Fig. 1. Normalized radiation resistance as a function of driving frequency for $f_0/f_{He} = 5$ and various values of ϕ_0, the angle of dipole orientation. In this plot, $f_{\mathrm{LHR}} \sim 2.3 \times 10^{-2} f_{He}$.

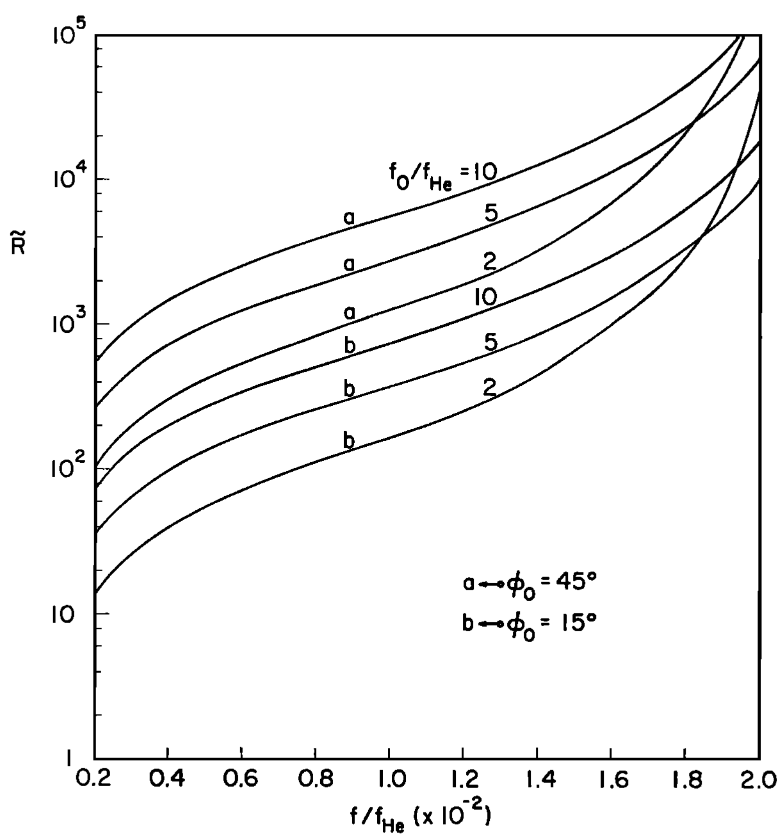

Fig. 2. Normalized radiation resistance as a function of driving frequency for f_0/f_{He} = 10, 5, 2, and two particular dipole orientations: $\phi_0 = 15°$ and $\phi_0 = 45°$.

the normalized radiation resistance $\tilde{R}$ is plotted against the normalized frequency f/f_{He} for various angles ϕ_o of antenna orientation with respect to the static magnetic field and for three different values of f_0/f_{He}. In all cases, an electron-proton plasma is assumed. From these numerical plots and from (7) and (12), we observe the following interesting properties of the radiation resistance: (1). $\tilde{R} \rightarrow \infty$ as $f \rightarrow f_{\text{LHR}}(f_{\text{LHR}} \sim 2.3 \times 10^{-2} f_{He})$. (2). For a fixed value of f_0/f_{He} and $f \neq f_{\text{LHR}}$, the radiation resistance increases rapidly as ϕ_0 varies from 0° to ~30° and gradually from 30° up to 90°. The ratio of $\tilde{R}_\perp/\tilde{R}_\parallel$ is approximately equal to $(f_{He}/f)^2$. (3). For a fixed angle of dipole orientation, the radiation resistance is proportional to f_0/f_{He} for values of frequency neither close to f_{LHR} nor close to f_{Hp}. On the other hand, if $f \sim f_{\text{LHR}}$, $\tilde{R}$ tends to increase as f_0/f_{He} decreases because of the shift in f_{LHR} to lower frequencies as f_0 decreases. (4). Except for the case of $\phi_0 \cong 0$ and $f < 10^{-2} f_{He}$, $\tilde{R}$ is from ~10^2 to ~10^5. This indicates that for the low frequency range considered, a dipole antenna in the magnetoplasma is much more efficient than it would be in free space. (5). In regard to the case $f \rightarrow f_{\text{LHR}}$, it should be noted that (22*a*) is independent of restriction (2) as $f \rightarrow f_{\text{LHR}}$, but (22*b*) is not independent of (9). Thus (22*b*) is useful only as long as (9) is satisfied.

The result that $R_\perp \gg R_\parallel$ may be qualitatively explained as follows: For the VLF frequency range considered, the dipole with parallel orientation primarily excites the ordinary mode wave, which is not a propagating mode. However, some power is still transmitted through the medium because of the secondary sources that result from the finite electrical compressibility of the plasma (i.e., $\nabla \cdot \mathbf{E} \neq 0$). On the other hand, the dipole with perpendicular orientation primarily excites two 'principal' wave modes, E_{+1} and E_{-1}, [*Wang and Bell*, 1969], which are propagating modes for the VLF range, and these two modes further excite all three 'principal' modes E_ν, $\nu = +1, -1, 0$ through the $\nabla \cdot \mathbf{E}$ term. Consequently, it might be expected that considerably more power will be radiated from the dipole oriented perpendicular to the magnetic field and a much higher radiation resistance achieved than in the case of a dipole in a parallel orientation.

In deriving (1), (8), and (16), we have neglected complicating effects due to finite plasma temperature, finite plasma dimensions, and plasma inhomogeneities and nonlinearities. Furthermore, we have not considered the presence of conducting or dielectric bodies in the vicinity of the antenna. However, if we assume that there are realistic circumstances under which these idealizations are acceptable, then the usefulness of the relations derived in this paper depend on satisfying the general restriction (15). In this case, it is worthwhile to give a few examples of the range of antenna lengths for which (15) can be satisfied in the magnetosphere. If we adopt the gyrofrequency model of electron density [*Helliwell,* 1965] and the centered dipole model of the earth's magnetic field, (15) can be reduced to the following inequality:

$$4h^2L^{-3} \ll 10^4[(f_{\mathrm{LHR}}^2/f^2) - 1] \text{ meters}^2 \qquad (23)$$

where L is the distance from the earth's center to the antenna location in units of earth radii.

From (23), it can be seen that for $L > 2$, (16) will hold for antennas whose total length is as long as 100 meters, given that $f \leq 1.4\, f_{\mathrm{LHR}}$. Furthermore, at $L = 3$, (16) would apply to a 100-meter antenna for frequencies in the approximate range $f_{Hp} \leq f \leq 1.16\, f_{\mathrm{LHR}}$. Finally, at $L = 2$ and $f \leq 10^{-1} f_{\mathrm{LHR}}$, (16) should hold for dipole antennas ≥ 1 km in length.

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