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Radiation resistance of a short dipole immersed in a cold magnetoionic medium[1]

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Electromagnetic radiation from sources immersed in a cold magnetoplasma is analyzed by the use of 'principal-polarized' wave coordinates; and the wave fields, the mean complex radiated power, and the Poynting vector are systematically expressed in terms of 'polarized' wave modes. For each polarized wave mode, the medium is effectively isotropic, since the Fourier-transformed wave equation for each principal mode is decoupled and is similar to the one obtained from the usual scalar Helmholtz operator. In application of the theory, consideration is made of the radiation resistance of a linear electric antenna of moderate length oriented both parallel and perpendicular to the static magnetic field. Approximate closed form expressions for the radiation resistance are obtained for the VLF frequency range by using a plasma model appropriate to the magnetosphere. These approximate closed-form expressions are compared with the results obtained previously by other workers using a quasi-static approximation, and it is shown that excellent agreement exists between the two methods of analysis. It is concluded that the quasi-static approximation can accurately predict the radiation resistance of a linear antenna in the magnetosphere, given certain moderate restrictions on the antenna length.

1. INTRODUCTION

In recent years, considerable attention has been devoted to the problem of electromagnetic radiation from sources immersed in a relatively cold magnetized plasma such as exists in the inner magnetosphere. The anisotropy and electrical compressibility of the magnetoionic medium make the study of electromagnetic radiation in this medium a mathematically complex and difficult one, but nevertheless it is a study that is being carried forward energetically as scientific needs for new plasma diagnostic tools and engineering needs for new satellite communication systems become more acute.

One area in which much more study is needed is that involving antenna radiation characteristics at very low frequency (VLF), where the anisotropy of the magnetized plasma is most pronounced. This need has been emphasized recently by the great interest on the part of the geophysical scientific community in the idea of a satellite-borne VLF transmitter operating in the magnetosphere. Such a transmitter would provide a means of performing a number of interesting and important experiments involving wave-particle interactions, wave propagation phenomena, and plasma diagnostics. The usefulness of this transmitter in performing experimental tasks will be limited ultimately by the amount of power that can be radiated into the plasma from the antenna (in some instances the radiation pattern will also play a critical role). In this connection, a knowledge of the coupling between the plasma and antenna is crucial.

It is the purpose of the present paper to attempt to give some insight into the problem of the coupling between a satellite VLF antenna system and the magnetospheric plasma by the consideration of some idealized cases. Specifically we calculate the radiation resistance of a thin electric dipole of moderate length oriented either parallel or perpendicular to the static magnetic field.

Our work differs from that of other workers who have considered similar or identical problems [*Bunkin*, 1957; *Kogelnik*, 1960*a, b*; *Kuehl*, 1962; *Mittra and Deschamps*, 1963; *Arbel and Felsen*, 1963; *Balmain*, 1964; *Ament et al.*, 1964; *Staras*, 1964; *Weil and Walsh*, 1964; *Blair*, 1964; *Seshadri*, 1965;

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Galejs, 1966*a, b*] in that we have started from a full-wave treatment of the problem and have derived approximate closed form expressions for the radiation resistance which are valid for a wide range of antenna lengths. Using these expressions, we are able to define for the first time the range of antenna length for which a quasi-static analysis, such as that employed by *Balmain* [1964] and *Blair* [1964], accurately predicts the radiation resistance of the antenna.

The plan of our paper is as follows: In section 2, we investigate the radiation problem by formulating it in a 'principal-polarized' wave coordinate system. With these coordinates, the inversion of the wave matrix becomes trivial, and the term that causes the medium to be electrically compressed, i.e., $\nabla \cdot \mathbf{E}$, can be easily and directly expressed in terms of external sources. For each principal-polarized wave mode, the medium is effectively isotropic, since the Fourier-transformed wave equation for each principal mode is decoupled by the proper expression of $\nabla \cdot \mathbf{E}$ in terms of external sources. The Fourier-transformed electric fields for each principal mode are then obtained with the characteristic poles of the system explicitly shown. The mean complex radiated power, thus the input reactance and radiation resistance, and the complex Poynting vector are also systematically expressed in terms of principal-polarized wave modes.

In section 3, in application of the present formulation, we consider the radiation resistance for two different orientations of a linear electric antenna with a triangular current distribution. The analysis has been made only in the VLF frequency range ($Y \geq 1$), where approximate closed-form expressions for the radiation resistance have been obtained under the two assumptions that the plasma parameters are appropriate to those of the magnetosphere ($X \gg Y^2$) and that the antenna is of moderate length. (The mathematical definition of the word 'moderate' can be found in equation 25, and discussion of the physical implications of the word is given in section 3.3.) These closed-form expressions for the radiation resistance are compared with the results obtained previously by other workers using a quasi-static approximation, and it is shown that excellent agreement exists between the two methods of analysis. Approximate 'necessary and sufficient' conditions for the validity of the quasi-static approximation are given in terms of restrictions on the antenna length; and some numerical examples, illustrating the physical meaning of the length restrictions, are computed by using the gyrofrequency model of the magnetosphere.

A comparison of the results of the present paper with the results of other workers in the field is made in section 4, and in section 5 a brief discussion of our results is given.

2. ANALYTICAL FORMULATION

In this section we give a general formulation in 'principal-polarized' wave coordinates for the problem of electromagnetic radiation with sources in cold magnetoplasma. The Fourier-transformed electric and magnetic fields, the mean complex radiated power (thus the input reactance and the radiation resistance), and the Poynting vector are all expressed in terms of the principal-wave modes.

2.1 *Field vectors* (**E**, **H**). The basic equations governing the electromagnetic radiation from a time harmonic ($e^{j\omega t}$) source immersed in a uniform, cold magnetoplasma are (in rmks units)

$$\nabla \times \mathbf{E} = -j\omega\mu_0\mathbf{H} \tag{1}$$

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} - |e|\, N_0\mathbf{v} + \mathbf{J} \tag{2}$$

$$(j\omega + c_e)\mathbf{v} = -\frac{|e|}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}_0) \tag{3}$$

where ϵ, N_0, $\mathbf{v}$, c_e, and $\mathbf{J}$ stand for the free-space permittivity, the unperturbed density, the ordered velocity of electrons, the effective electron collision frequency, and the external current, respectively. Substituting (2) into the curl of (1) gives

$$\nabla^2\mathbf{E} + \beta^2\mathbf{E} - \nabla(\nabla \cdot \mathbf{E}) = \frac{j\beta^2}{\omega\epsilon}(\mathbf{J} - |e|\, N_0\mathbf{v}) \tag{4}$$

where $\beta = \omega/c$, the wave number in free space. By applying the three-dimensional Fourier space transform to (4), a transformed wave equation can be written

$$k^2\vec{\mathcal{E}} - \beta^2\mathcal{E} - \mathbf{k}(\mathbf{k}\cdot\vec{\mathcal{E}}) = \frac{\beta^2}{j\omega\epsilon}(\vec{\mathcal{J}} - |e|\, N_0\vec{\mathcal{V}}) \tag{5}$$

where $\vec{\mathcal{E}}$, $\vec{\mathcal{J}}$, and $\vec{\mathcal{V}}$ are the Fourier-transformed electric fields, external current, and electron-ordered velocity, respectively.

To transform (5) into polarized-wave coordinates, we define a set of polarized phasors

$$\mathcal{E}_{\pm 1} = \frac{1}{(2)^{1/2}}(\mathcal{E}_x \pm j\mathcal{E}_y), \quad \left.\begin{matrix}\text{right}\\ \text{left}\end{matrix}\right\}\text{elliptically polarized}$$

$$\mathcal{E}_0 = \mathcal{E}_z, \text{ longitudinally polarized} \tag{6}$$

with similar definitions for k_ν, $\mathcal{J}_\nu$, and $\mathcal{V}_\nu$ ($\nu = +1, -1, 0$). The inversions of these quantities follow directly from (6).

With the above definitions, the wave equation (5) can be written in polarized-wave coordinates ($\nu = +1, -1, 0$ coordinates)

$$k^2 \mathcal{E}_\nu - \beta^2 \vec{\mathcal{E}}_\nu - k_\nu(\mathbf{k}\cdot\vec{\mathcal{E}}) = \frac{\beta^2}{j\omega\epsilon}(\mathcal{J}_\mu\, \delta_{\nu\mu} - |e|\, N_0 \mathcal{V}_\nu) \quad (7)$$

where $\delta_{\nu\mu} = 1$, when $\nu = \mu$ and zero otherwise.

Orienting the spatial coordinates so that $\mathbf{B}_0 = \hat{z}B_0$, we can use (3) to obtain $\mathcal{V}_\nu$

$$\mathcal{V}_\nu = \frac{-|e|\ \mathcal{E}_\nu}{jm(\omega - jc_e - \nu\Omega_0)} \quad (8)$$

where $\Omega_0 = (|e|B_0/m)$, the magnitude of electron gyrofrequency.

Substituting (8) back into (7) and rearranging the result gives

$$\mathcal{E}_\nu - \frac{k_\nu}{k^2 - \beta^2\epsilon_\nu}(\mathbf{k}\cdot\vec{\mathcal{E}}) = \frac{\beta^2 \mathcal{J}_\mu}{j\omega\epsilon(k^2 - \beta^2\epsilon_\nu)}\,\delta_{\nu\mu} \quad (9)$$

where $\epsilon_\nu = 1 - (X/1 - jZ - \nu Y)$, the principal relative dielectric permittivity, and X, Z, and Y are standard notations for normalized frequencies in magnetoionic theory [*Ratcliffe*, 1959].

Using (9) in conjunction with the identity $\mathbf{k}\cdot\vec{\mathcal{E}} = \sum k_{-\nu}\mathcal{E}_\nu$, the term which causes the medium to be electrically compressed, i.e., $\mathbf{k}\cdot\vec{\mathcal{E}}$, can be directly expressed in terms of external sources

$$\mathbf{k}\cdot\vec{\mathcal{E}} = \frac{j}{\omega\epsilon}\,\frac{\prod(k)}{\alpha(\theta)(k^2 - k_+^2)(k^2 - k_-^2)} \sum_\mu \frac{k_{-\mu}\mathcal{J}_\mu}{(k^2 - \beta^2\epsilon_\mu)}, \qquad \mu = +1, -1, 0 \quad (10)$$

where

$$\prod(k) = (k^2 - \beta^2\epsilon_{+1})(k^2 - \beta^2\epsilon_{-1})(k^2 - \beta^2\epsilon_0)$$

$$\alpha(\theta) = \epsilon_s \sin^2\theta + \epsilon_0 \cos^2\theta, \qquad \epsilon_s = \tfrac{1}{2}(\epsilon_{+1} + \epsilon_{-1})$$

and

$$k_\pm^2 = \frac{-\beta^2[(\epsilon_s\epsilon_0 - \epsilon_{+1}\epsilon_{-1})\sin^2\theta - 2\epsilon_0\epsilon_s] \pm \beta^2[(\epsilon_s\epsilon_0 - \epsilon_{+1}\epsilon_{-1})^2\sin^4\theta + \epsilon_0^2(\epsilon_{-1} - \epsilon_{+1})^2\cos^2\theta]^{1/2}}{2\alpha(\theta)} \quad (11)$$

Substituting (10) into (9) gives the Fourier-transformed electric fields for each polarized wave

$$\mathcal{E}_\nu = \frac{1}{j\omega\epsilon}\left[\frac{\beta^2\mathcal{J}_\mu}{(k^2 - \beta^2\epsilon_\nu)}\,\delta_{\nu\mu} - \frac{\prod(k)}{\alpha(\theta)(k^2 - k_+^2)(k^2 - k_-^2)}\,\frac{k_\nu}{(k^2 - \beta^2\epsilon_\nu)} \cdot \sum_\mu \frac{k_{-\mu}\mathcal{J}_\mu}{(k^2 - \beta^2\epsilon_\mu)}\right] \quad (12)$$

The first term on the right-hand side of (12) is caused by the external source alone ($\delta_{\nu\mu}$ implies that $\mathcal{J}_\mu$ excites only $\mathcal{E}_\mu$), whereas the second term is due to the finite electrical compressibility.

Using the Fourier transform of (1) and the definitions (6), the Fourier-transformed magnetic fields for each principal polarized mode can be found:

$$\vec{\mathcal{H}} = \frac{1}{j\omega\mu_0}\begin{vmatrix} \hat{U}_{-1} & \hat{U}_0 & \hat{U}_{+1} \\ k_{+1} & k_0 & k_{-1} \\ \mathcal{E}_{+1} & \mathcal{E}_0 & \mathcal{E}_{-1} \end{vmatrix} \quad (13)$$

where $\hat{U}_{\pm 1} = 1/(2)^{1/2}(\hat{x} \pm j\hat{y})$, $\hat{U}_0 = \hat{z}$, and $\hat{x}$, $\hat{y}$ and $\hat{z}$ are the unit vectors in Cartesian coordinates.

2.2 *General form for radiation resistance, input reactance, and Poynting vector.* In terms of polarized-wave coordinates, the mean complex power radiated by the current distribution $\mathbf{J}(\mathbf{x})$, can be written

$$P = -\frac{1}{2}\int \sum_\nu J_\nu^*(\mathbf{x})E_\nu(\mathbf{x})\, d\mathbf{x} \quad (14)$$

Using Parseval's theorem, we can express the mean complex power in $\mathbf{k}$ space [*Sneddon*, 1951]:

$$P = \frac{-1}{16\pi^3}\int \sum_\nu \mathcal{J}_\nu^*(\mathbf{k})\,\mathcal{E}_\nu(-\mathbf{k})\, d\mathbf{k} \quad (15)$$

The radiation resistance and input reactance are related simply to the real and imaginary parts of the complex power

$$R = \frac{2\,\mathrm{Re}\,P}{I_0^2}, \qquad \chi = \frac{2\,\mathrm{Im}\,P}{I_0^2} \quad (16)$$

where I_0 is the effective terminal current of the antenna.

In terms of the polarized fields, the time averaged complex Poynting vector can be expressed by the following determinant:

$$\mathbf{S} = \frac{j}{2}\begin{vmatrix} \hat{U}_{+1} & \hat{U}_0 & \hat{U}_{-1} \\ E_{-1} & E_0 & E_{+1} \\ H_{+1}^* & H_0^* & H_{-1}^* \end{vmatrix} \quad (17)$$

and the Cartesian components of $\mathbf{S}$ are given by

$$S_x = \frac{j}{2(2)^{1/2}} [E_0(H_{-1}^* - H_{+1}^*) - H_0^*(E_{+1} - E_{-1})]$$

$$S_y = \frac{j}{2(2)^{1/2}} [E_0(H_{+1}^* + H_{-1}^*) - H_0^*(E_{+1} + E_{-1})] \quad (18)$$

$$S_z = \frac{j}{2} [E_{+1}H_{+1}^* - E_{-1}H_{-1}^*]$$

3. RADIATION RESISTANCE FOR TWO PARTICULAR ORIENTATIONS OF AN ELECTRIC DIPOLE

In this section, we make an application of the polarized wave formulation by computing an approximate closed-form expression for the radiation resistance of a finite linear electric antenna oriented either parallel or perpendicular to the static magnetic field. The current distribution on the antenna is assumed to be triangular. Since our main interest involves the VLF characteristics of linear electric antennas in the magnetosphere, we have focused our attention on the plasma parameters most appropriate to the magnetosphere ($X \gg Y^2$; Ratcliffe notation) and to frequencies in the VLF range ($Y > 1$). In the VLF range in the magnetosphere, for frequencies higher than the proton gyrofrequency, it is known that in general the only propagating electromagnetic mode is the whistler mode [corresponding to k_- in (11)], and consequently only this mode can contribute to the radiation power from a VLF antenna. An interesting feature of the whistler mode waves in the cold plasma theory is the fact that the waves are propagating only when the angle between the wave normal and the static magnetic field lies within the range $0 \le \theta \le \theta_r = \tan^{-1} (-\epsilon_0/\epsilon_R)^{\frac{1}{2}}$, and in this angular range the wave number k_- increases monotonically with θ, approaching infinity as $\theta \to \theta_r$. This behavior of the wave number as a function of angle precludes the possibility of uniquely classifying any given VLF antenna as a 'short' antenna, since no matter how short it might be physically, it is always possible to find some wave normal angle θ for which the antenna length in that direction will exceed one wavelength in the medium and appear to be 'long.'

On the other hand, the total radiation resistance involves an integral over θ, and it is possible that a given antenna may appear to be 'short' over a large enough portion of the range $0 \le \theta \le \theta_r$, so that its primary integrated behavior is that of a short antenna.

This fact proves to be the case for moderate-length VLF satellite antennas in most regions of the inner magnetosphere ($1.5 < L < 4$) and forms the basis for our treatment of the integrals in the following analysis.

3.1 *Parallel orientation.* We center the dipole antenna of length $2h$ in a dextral Cartesian coordinate system parallel to $\mathbf{B}_0$. The antenna current is assumed to be

$$\begin{aligned} J_0 &= I_0\left(1 - \frac{|z|}{h}\right) \delta(x)\, \delta(y), \qquad |z| \le h \\ &= 0, \qquad |z| > h \qquad (19) \\ J_{+1} &= J_{-1} = 0 \end{aligned}$$

The Fourier transform of (19) is

$$\begin{aligned} \mathcal{J}_0 &= \frac{4I_0}{h} \frac{\sin^2 (hk_z/2)}{k_z^2} \\ \mathcal{J}_{+1} &= \mathcal{J}_{-1} = 0 \end{aligned} \qquad (20)$$

For this case, (15) becomes

$$P = -\frac{1}{16\pi^3} \int \mathcal{J}_0^*(\mathbf{k})\, \mathcal{E}_0(\mathbf{k})\, d\mathbf{k} \qquad (21)$$

Using $\mathcal{E}_0$ from (12) together with (20) and (21), we have the following integral for the complex power in spherical **k**-space coordinates:

$$P_\parallel = C_0 \int_0^\infty \int_0^\pi \int_0^{2\pi} \left[\frac{\beta^2 \cos^{-4} \theta}{k^2(k^2 - \beta^2\epsilon_0)} - \frac{(k^2 - \beta^2\epsilon_{+1})(k^2 - \beta^2\epsilon_{-1}) \cos^{-2} \theta}{\alpha(\theta)(k^2 - k_+^2)(k^2 - k_-^2)(k^2 - \beta^2\epsilon_0)}\right] \cdot\sin^4 \left(\frac{hk \cos \theta}{2}\right) \times \sin \theta \, d\psi \, d\theta \, dk \qquad (22)$$

where $C_0 = (jZ_0I_0^2/\pi^3h^2\beta)$, $Z_0 = (\mu/\epsilon)^{\frac{1}{2}} \cong 377$ ohms, and $\alpha(\theta)$ is defined in (11).

To evaluate (22), we perform the trivial ψ integration and then make use of the fact that the integrand is an even function of k to extend the k integration along the entire real k axis. The k integration can then be performed by using the technique of contour integration.

In performing the contour integration, we assume the medium to be slightly 'lossy' to ensure convergence of the integral and split the fourth-power sine function into a combination of exponentials according to the relation $\sin^4 x = \frac{1}{16} [(\exp jx - \exp - jx)]^4$. We then close the contour in the lower-half k plane for all terms involving a negative exponential exp $(-jx)$, and close the contour in the upper-half k plane for all terms involving a positive exponential exp (jx), taking care to note that the real and imaginary parts of k_- have opposite signs. As a final step, the losses are allowed to approach zero.

In evaluating (22) for the VLF frequency range, it is found that only the pole k_- contributes to the real radiated power, and the following expression is obtained for the radiation resistance of the parallel antenna:

$$R_\parallel = \frac{Z_0}{4\pi}(h\beta)^2 \int_0^{\theta_r} \frac{n^3(n^2 - \epsilon_{+1})(n^2 - \epsilon_{-1})}{(n^2 - \epsilon_0)G(\theta)} \left(\frac{\sin \lambda n}{\lambda n}\right)^4 \cdot \cos^2 \theta \sin \theta \, d\theta \qquad (23)$$

where $n = k_-(\theta)/\beta$, $\theta_r = \tan^{-1}(|\epsilon_0/\epsilon_s|)^{1/2}$, $\lambda = (h\beta/2)\cos\theta$, $G(\theta) = \alpha(\theta)(k_+^2 - k_-^2)/\beta^2$, and we have used (16) to evaluate $R_\parallel$ from $P_\parallel$.

Equation 23 can be accurately approximated by the first few terms of a power series in h given the condition

$$\gamma_m^2 = n^2(\theta_m)\cos^2\theta_m \ll (h\beta)^{-2}; \qquad \theta_m < \theta_r \qquad (24)$$

where θ_m is the angle at which the quantity $n \cos \theta$ has its minimum value over the range $0 \le \theta < \theta_r$, and 'γ_m' is by definition this minimum value.

Using (11) to find θ_m, (24) can be rewritten as the more definitive conditions

$$\gamma_m^2 = \left[\frac{(1-a)^{1/2} + (1-\epsilon_0)^{1/2}}{(1-\epsilon_0)^{1/2} - (1-a)^{1/2}}\right] \ll (h\beta)^{-2}; \qquad Y > 2\epsilon_0/(\epsilon_0 + 1) \qquad (25a)$$

$$\gamma_m^2 = \epsilon_{+1} \ll (h\beta)^{-2}; \qquad 1 \le Y \le \frac{2\epsilon_0}{\epsilon_0 + 1} \qquad (25b)$$

where $a = (\epsilon_{+1}\epsilon_{-1})/\epsilon_s$.

The details of the approximation of (23) are presented in appendix A. Below we give the result for the situation in which $X \gg Y^2$, a condition which holds throughout large regions of the magnetosphere

$$R_\parallel = \frac{Z_0}{2\epsilon_s(h\beta)}\left[1 - \frac{2\pi + 1}{6\pi}\epsilon_s(h\beta)^2 - \frac{2\pi}{3}\left(\frac{X}{Y-1}\right)^{1/2}\frac{Y-2}{Y^3}(h\beta)^3\right] \qquad (26)$$

where $\epsilon_s \cong X/(Y^2 - 1)$, and $X \gg Y^2$.

Evaluating (25) for the case $X \gg Y^2$ leads to the following result concerning the range of h for which (26) is valid:

$$(h\beta)^2 \ll Y^2/4X; \qquad Y \ge 2$$
$$(h\beta)^2 \ll (Y-1)/X; \qquad 2 \ge Y \ge 1 \qquad (27)$$

The restriction on h as given in (27) formally ensures that the power series of (A5) will be rapidly convergent, but it is too stringent for practical requirements. It is possible to show that we need only require the condition

$$(h\beta\gamma_m)^2 \le 1/4 \qquad (28)$$

in order to be able to approximate $R_\parallel$ to within a few per cent using (26).

3.2 PERPENDICULAR ORIENTATION

For the case of the linear antenna oriented perpendicular to the static magnetic field, it is necessary to introduce a finite radius for the antenna in order to secure the convergence of the integral in (15). In this case the current distribution takes the form

$$J_{\pm1} = \frac{I_0}{(2)^{1/2}}\left(1 - \frac{|x|}{h}\right)\frac{\delta(r_\perp - b)}{2\pi}, \qquad |x| \le h$$
$$J_{\pm1} = 0, \qquad |x| > h \qquad (29)$$
$$J_0 = 0$$

where $r_\perp = (y^2 + z^2)^{1/2}$, b = antenna radius, and the antenna is assumed to be aligned parallel to the x axis.

Equation 29 leads to the Fourier-transformed currents

$$\mathcal{J}_{\pm1} = \frac{4I_0}{(2)^{1/2}h}\frac{\sin^2(hk_x/2)}{k_x^2}J_0(bk_\perp)$$
$$\mathcal{J}_0 = 0 \qquad (30)$$

where $k_\perp = (k^2 - k_x^2)^{1/2}$.

Formally, (15) can now be written

$$P_\perp = \frac{-1}{16\pi^3}\int(\mathcal{E}_{+1} + \mathcal{E}_{-1})\mathcal{J}_{+1}\, d\bar{k} \qquad (31)$$

where the field quantities $\mathcal{E}_{\pm1}$ are to be calculated using (12) and (30). In evaluating (31) our interest lies mainly in those cases in which the antenna length is great enough so we can always neglect any terms of the order (b/h) which appear in the power series for $P_\perp$. Thus, in effect, we wish to find the limiting form of $P_\perp$ as 'b' → 0. An approximate evaluation of the real part of (31) (for the limiting case 'b' → 0) is performed in appendix B; the result follows:

$$R_\perp \cong \frac{Z_0\zeta^{-1}}{\pi\alpha(|\epsilon_0|)^{1/2}} \cdot \left[\ln\left(\frac{2h}{b\alpha}\right) - 1 - \frac{1}{4}\zeta^2 + \frac{(Y+1)^{3/2}}{12}\zeta^3\right] \qquad (32)$$

where

$$\zeta = (h\beta/\alpha)\epsilon_s^{1/2}, \qquad \epsilon_s \cong X/(Y^2 - 1),$$

$$\alpha = [(\epsilon_0 - \epsilon_s)/\epsilon_0]^{1/2} \approx [Y^2/(Y^2 - 1)]^{1/2},$$

and it is assumed that

$$(h\beta)^2 \leq \frac{\alpha^2(Y^2 - 1)^{1/2}}{8X} \tag{33}$$

In (33) the formal constraint (B6) has been relaxed for practical purposes, and this yields an approximate over-all error in (32) of less than 10%.

3.3 *Region of validity of the quasi-static approximation.* As long as (28) is observed, the first term of (26) is by far the largest in the series over the entire range of $Y(\infty > Y \geq 1)$ and represents at least 9/10 of the series sum. Thus to within 10%

$$R_\parallel \approx R_\parallel^q = (Z_0/2\epsilon_s h\beta) \tag{34}$$

It is interesting to note that (34) is identical to the result that has been obtained by *Balmain* [1964] and other workers using a quasi-static analysis of the same antenna configuration. This close agreement indicates that the quasi-static approximation is quite accurate for calculating the real part of the complex antenna impedance, given the assumptions leading up to (26), and it is of interest to establish in a more general way the conditions under which it can be expected that a quasi-static analysis will furnish a good approximation to the radiation resistance of the parallel antenna. This condition can be deduced by noting that an upper bound for $R_\parallel$ is provided by the expression

$$R_m = \frac{Z_0}{2\pi\epsilon_s}(h\beta)^2 \int_{\gamma_m}^{\infty} \gamma^2\left(\frac{\sin \lambda\gamma}{\lambda\gamma}\right)^4 d\gamma \tag{35}$$

which we have obtained by replacing both R_+ and R_- in (A3) by the factor $2\gamma^2$ and by extending the range of integration of the first integral in (A3) up to infinity. Since (35) represents a 'generous' upper bound to $R_\parallel$, it is clear that the quasi-static approximation will fail to describe $R_\parallel$ accurately whenever $R_\parallel^q \geq R_m$.

The integral contained in (35) has been evaluated by *Seshadri* [1965], and, using his result, the condition under which $R_\parallel^q \cong R_m$ can be established through a numerical analysis. It is found that $R_\parallel^q \cong R_m$ when $\lambda\gamma_m \sim \pi/2$, and thus it can be stated that a necessary condition for the validity of the quasi-static approximation (as applied to the parallel antenna described above) is that

$$h\beta\gamma_m < \pi \tag{36}$$

In the light of (36) it is clear that (28) represents roughly both a necessary and sufficient condition for the validity of the quasi-static approximation, and one physical intepretation of the restriction of (28) is that the quasi-static approximation will be valid for the parallel antenna so long as some direction in space can be found for which the antenna appears 'short' and for which all portions of the antenna radiate approximately in phase.

As for the case of the perpendicular antenna, it is interesting to compare (32) with the results obtained by *Balmain* [1964] and others for the perpendicular antenna using the quasi-static approximation. Using Balmain's results in conjunction with our own notation, the following expression is obtained:

$$R_\perp^q = \frac{Z_0\zeta^{-1}}{\pi\alpha(|\epsilon_0|)^{1/2}}\left[\ln\frac{2h}{b\alpha} - 1\right] \tag{37}$$

Equation 37 shows that the quasi-static 'radiation resistance' is exactly the same as the first two terms of (32), and it is clear that these first terms represent the major portion of (32). Thus our results compare closely with those of the quasi-static approximation. As in section 3.1, it can be shown by using arguments based on upper bounds that the condition for the validity of the quasi-static analysis is given approximately by (33).

3.4 *Physical interpretation of antenna constraints.* Equations 28 and 33 give the mathematical constraints on the antenna length under which (26) and (32) are good approximations to the total radiated power. To determine if these constraints lead to situations of any possible physical significance in the magnetosphere, it is necessary to consider these constraints in the light of physically acceptable models of the magnetospheric plasma.

One reasonable and simple model of the inner magnetosphere ($L < 4$) is the so-called 'gyrofrequency' model [*Helliwell*, 1965], in which the plasma electron density at any point is proportional to the earth's dipole magnetic field strength at that point. In this model the relationship between the plasma parameters X and Y has the form

$$X = (A/f_H)Y^2 \tag{38}$$

where f_H is the electron gyrofrequency and A is a constant. The work of *Smith* [1960] can be used to obtain a value of A that is characteristic of a solar maximum period, i.e., $A \approx 10^6$ Hz, and the variation of f_H can be obtained from the dipole field approximation [*Helliwell,* 1965]:

$$f_H \approx 9 \times 10^5 L^{-3}(1 + 3 \sin^2 \phi)^{1/2} \quad \text{Hz} \qquad (39)$$

where L is the distance from the earth center measured in earth radii and ϕ is the magnetic latitude.

Equations 38 and 39 can be used in conjunction with (28) to yield the following constraint on the antenna length for the case of the parallel antenna:

$$\begin{aligned} h^2 &\ll 6 \times 10^2 Y^2 L^3 (\text{meters})^2; \qquad Y \geq 2 \\ h^2 &\ll 2.4 \times 10^3 (Y - 1) L^3; \qquad 1 \leq Y \leq 2 \end{aligned} \qquad (40)$$

where the latitudinal variation of the magnetic field strength has been neglected.

A few simple numerical examples serve to illustrate the content of (40). For instance if the transmitted signal is to be 5 kHz at $L = 3$ (where $Y \sim 6$), h must be such that $h^2 \ll 6 \times 10^5$ m^2 and (26) will be a good approximation for antenna lengths ($2h$ = total length) up to about 500 meters. At $L = 2$ and 5 kHz ($Y \sim 20$), (26) will be a good approximation for antenna lengths up to about 1.5 km. In the general case, as long as $Y > 2$ and $L \geq 2$, (26) will be valid for antenna lengths up to at least 100 meters. However, the usefulness of the expression is severely limited as the wave frequency approaches the gyrofrequency ($Y \to 1$), since only antennas of vanishingly small length can then satisfy (40).

A somewhat different situation occurs in the case of the perpendicular orientation, since (33) shows that in the limit as $Y \to 1$ the antenna length can become arbitrarily large and still satisfy the constraint. This behavior is also true in the limit $\omega \to 0$ ($Y \to \infty$) as is evident in the form obtained after using (38) and (39) in (33)

$$h^2 \ll 1.2 \times 10^3 L^3 Y^2/(Y^2 - 1)^{1/2} \text{ m}^2 \qquad (41)$$

where again the latitudinal variation of the magnetic field has been neglected.

The right-hand side of (41) possesses a minimum value as a function of Y, at the point $Y = (2)^{1/2}$, and it is convenient to use this as a reference value to determine the range of h which will satisfy (32) for all Y. Thus at $Y = (2)^{1/2}$, (41) becomes $h^2 \ll 2.4 \times 10^3 L^3$ m^2, and it can be stated that, for all values of $Y \geq 1$, (32) will be a good approximation for antenna lengths up to about 200 meters, given that $L \geq 2$. For any other value of Y, keeping L fixed, (32) will be valid for linear antennas longer than 200 meters, as determined by (41).

The above examples serve to illustrate the fact that in general our expressions for $R_\parallel$ and $R_\perp$ will be valid for VLF antenna lengths of the order of, or greater than, those presently being employed in magnetospheric satellite experiments, and thus these expressions possess a practical worth in providing insight into the coupling between satellite VLF antenna systems and the magnetospheric plasma.

4. COMPARISON OF RESULTS

A number of other workers have treated the problem of VLF radiation characteristics of a thin dipole antenna of finite length in a homogeneous, cold magnetoplasma. These workers can be divided into two groups on the basis of whether a full-wave treatment or a quasi-static treatment was used in their paper. Without any exception known to the authors, those who have used a full-wave treatment [*Seshadri,* 1965; *Staras,* 1964; *Galejs,* 1966*a, b*] have presented their results on the radiation resistance in numerical form and have not given analytic closed-form results such as derived above, whereas those who have used a quasi-static approximation [*Balmain,* 1964; *Blair,* 1964] have obtained closed-form expressions for which no precise region of validity has been specified. The following is a comparison of our results with those of the aforementioned workers:

4.1 *Quasi-static treatment.* By the use of a quasi-static approximation, *Balmain* [1964] and *Blair* [1964] have analytically obtained the input impedance for a short, linear antenna with a triangular current distribution. For the VLF case with either a parallel or perpendicular antenna orientation, their closed-form expressions for the radiation resistance agree exactly with the leading terms of our own expressions (26) and (32) (given that the antenna is considered to be thin), and in the limit $h\beta \to 0$, our closed-form expressions agree in full.

Neither Balmain nor Blair specify the precise region of validity of their quasi-static approximation, indicating only that their results should hold when the antenna length is much less than the free-space wavelength at the driving frequency. However arguments presented in section 3 show that this criterion is not always correct and that the quasi-static approximation will be accurate only so long as the antenna length conforms to the constraints given in (28) and (33).

4.2 *Full-wave treatment.* *Seshadri* [1965] considers a current filament of finite length oriented parallel to the ambient magnetic field direction and possessing a triangular current distribution; thus this source is identical to that hypothesized in section 3.1 of the present paper, and it should be expected that our results would agree closely with his. This proves to be the case, and the VLF portion of Seshadri's Figure 4A showing radiation resistance versus frequency can be duplicated by using (26). Furthermore Seshadri's numerical results indicate that the radiation resistance at VLF increases as the length of current element decreases, which is in general agreement with our closed-form result.

Staras [1964] has calculated numerically the radiation resistance for the case of a finite dipole oriented either parallel or perpendicular to the ambient magnetic field, but his unusual choice of current distribution and limited numerical results makes a meaningful comparison between our results and his results extremely difficult. However, in a limited way, the general trend of Stara's results for the parallel antenna agrees with our results in that his radiation resistance varies roughly inversely as the antenna length for antennas between 30 and 300 meters long.

Another two papers in the field have been written by *Galejs* [1966*a*, *b*], who formulated a variational expression for the input impedance of a finite insulated cylindrical antenna with a longitudinal magnetic field and a finite insulated strip antenna with a perpendicular magnetic field. In both papers the results concerning radiation resistance are presented in numerical form, with the exception that in the low-frequency limit approximate, closed-form expressions are given for the quasi-static fields. In his papers, Galejs has compared his results with those predicted by the quasi-static approximation of *Balmain* [1964] and found that there was reasonable agreement at low frequency. Since our own results agree closely with the quasi-static approximation (as already discussed in section 3), it can be inferred that reasonable agreement also exists between Galejs' results and those of the present paper.

5. CONCLUSIONS

In the present paper we have analyzed the full-wave problem of electromagnetic radiation from sources in a cold magnetoplasma through the use of principal-polarized wave coordinates. In applying the resultant formulation, we have considered the radiation resistance of a filamentary electric dipole of finite length, possessing a triangular current distribution and oriented either parallel or perpendicular to the ambient magnetic field. Assuming plasma parameters appropriate to the VLF range in the magnetosphere ($X \gg Y^2$; $Y \geq 1$), we have obtained approximate closed-form expressions for the radiation resistance that are valid for a wide range of antenna lengths.

These closed-form expressions are useful not only because of their obvious utilitarian advantages over purely numerical results, but also (and most important) because they allow a reasonably precise check to be made on the closed-form results of approximate theories such as that of the quasi-static analysis. Thus in comparing our closed-form results with the closed-form results of the quasi-static approximation, we have been able for the first time to specify the range of antenna lengths for which the quasi-static approximation is valid. Our findings indicate that, for the particular VLF antenna configurations considered, the quasi-static approximation should be accurate for a wide range of antenna lengths in the magnetosphere, up to the order of kilometers, as discussed in section 3.4.

Since the quasi-static theory is relatively uncomplicated compared with the full-wave theory, there is reason to expect that the quasi-static theory can be used to good advantage in attacking more realistic and more complicated antenna problems. However, before the quasi-static approximation can be used with confidence in treating more realistic problems, it is necessary to define more explicitly the conditions under which this approximation is valid in a magnetoplasma. In this regard, the specification which we have made in the present paper of the realm of validity of the quasi-static approximation can be looked upon as a first step toward a more general appraisal of this theory.

APPENDIX A: EVALUATION OF $R_{\parallel}$

Equation 23 can be accurately approximated in closed form by the first few terms of a power series in $h\beta$, given that (25) holds. The first step is to express (23) in terms of a new variable γ such that

$$\gamma = n(\theta) \cos \theta \qquad \text{(A1)}$$

A relation between the differential elements $d\gamma$

and $d\theta$ can be obtained through use of (11) and (A1):

$$G^{-1}(\theta) \cos\theta \sin\theta \, d\theta = \pm\frac{1}{2\epsilon_s}[\xi(\gamma)\, U_\pm(\gamma)]^{-1}\gamma \, d\gamma \quad \text{(A2)}$$

where

$$U_\pm(\gamma) = 1 + \eta(\gamma) \pm \xi(\gamma)$$

$$\eta(\gamma) = \frac{1}{2}\left(a + \epsilon_0 - 2 + \frac{\epsilon_s - \epsilon_0}{\epsilon_s}\gamma^2\right)$$

$$\xi(\gamma) = [\eta^2(\gamma) - (a - 1)(\epsilon_0 - 1)]^{1/2}$$

Equations A1 and A2 then lead to the following form for (23):

$$R_l = \frac{Z_0}{8\pi\epsilon_s}(h\beta)^2\left[\int_{\gamma_m}^{(\epsilon_{+1})^{1/2}} R_-(\gamma)\left(\frac{\sin\lambda\gamma}{\lambda\gamma}\right)^4 d\gamma + \int_{\gamma_m}^{\infty} R_+(\gamma)\left(\frac{\sin\lambda\gamma}{\lambda\gamma}\right)^4 d\gamma\right] \quad \text{(A3)}$$

where $\lambda = (h\beta/2)$, γ_m = minimum value of γ (as given by the left-hand side of (25)), and

$$R_\pm(\gamma) = \frac{\epsilon_0 - \epsilon_s}{\epsilon_0}\gamma^2 \frac{[U_\pm(\gamma) - 1][U_\pm(\gamma) - \gamma^2]}{\xi(\gamma)[U_\pm(\gamma) - \epsilon_0]}$$

Consider the second integral in (A3); this integral contains the major contribution to the radiated power. To approximate this integral, we first split it into two parts in the following way:

$$\int_{\gamma_m}^{\infty} = \int_{\gamma_m}^{(h\beta)^{-1}} + \int_{(h\beta)^{-1}}^{\infty} = I_1 + I_2 \quad \text{(A4)}$$

In view of (25), the first integral I_1 involves those values of γ for which the antenna will appear 'short,' and we make the approximation here that the sine function in the integral can be replaced by its argument (i.e. $\sin\lambda\gamma \sim \lambda\gamma$). The second integral I_2 involves those values of γ for which the antenna appears 'long,' and, in view of (25), γ should be large enough in this range so that $R_+(\gamma)$ may be expanded in a rapidly convergent power series in inverse powers of γ. It is sufficient for our purposes to use just the first two terms of this series as shown below:

$$R_+(\gamma) \cong 2\left[\gamma^2 - \left(\frac{Y^2 + 1}{Y^2}\right)\epsilon_s\right] \quad \text{(A5)}$$

The foregoing approximation of the integral I_2 follows closely a method previously used by *Seshadri* [1965]; however, we deviate somewhat from his approach by using the following approximate relations to complete the integration of the terms in I_2:

$$\int_{1/2}^{\infty} z^{2m}\left(\frac{\sin z}{z}\right)^4 dz \cong \int_0^{\infty} z^{2m}\left(\frac{\sin z}{z}\right)^4 dz - \int_0^{1/2} z^{2m}\, dz \cong \pi/3 + m - (1/2)^{2m+1}/(2m + 1); \quad (m = 0, 1) \quad \text{(A6)}$$

The approximation in (A6) involves replacing the sine function by its argument in the second integral on the right-hand side. The use of this approximation is in keeping with our treatment of the integral I_1 of (A4). Now consider the first integral in (A3). If (25*b*) holds, $\gamma_m = (\epsilon_{+1})^{1/2}$ and the integral is identically equal to zero. If (25*a*) holds, the integral is not identically equal to zero, but, unless $a \sim \epsilon_0$ and $Y^2 \gg 1$, it will be the case that $(\epsilon_{+1})^{1/2} \leq (h\beta)^{-1}$, and the small argument approximation can be used over the entire range of the integral.

Finally, even if $a \sim \epsilon_0$ and $Y^2 \gg 1$, it is possible to show that the major contribution to the integral comes from the range of γ for which the small argument approximation is valid, so long as $\gamma_m \leq 1/4\,(h\beta)^{-1}$. In view of the above considerations, plus the fact that the integral will contribute only to the smallest terms retained in our series, it is sufficient for our purposes to use the small argument approximation over the entire range of γ in the first integral in (A3).

Using (A5) and (A6) in (A3), as well as the small argument approximations discussed above, the expression for the total radiated power becomes

$$R_l = \frac{Z_0}{4\pi\epsilon_s}\left[\left(\frac{6\pi - 1}{3}\right)(h\beta)^{-1} - \left(\frac{2\pi - 3}{3}\right)\epsilon_s \frac{Y^2 + 1}{Y^2}(h\beta) + F(h)\right] \quad \text{(A7)}$$

where

$$F(h) = (h\beta)^2\left[\int_{\gamma_m}^{(\epsilon_{+1})^{1/2}} R_-(\gamma)\, d\gamma + \int_{\gamma_m}^{(h\beta)^{-1}} R_+(\gamma)\, d\gamma\right]$$

In general the function $F(h)$ can be evaluated exactly in terms of elliptic function, and the resultant expression can be expanded in powers of $(h\beta)$. The retention of terms through order $(h\beta)^2$ furnishes the desired expression.

It can be deduced that the leading term of $F(h)$ will have the form $1/3(h\beta)^{-1}$; the magnitude of the other terms in $F(h)$ will depend upon the values of the parameters a, ϵ_0, and ϵ_{+1} in a more or less complicated fashion. In the present development we intend to calculate $F(h)$ only for those situations where $X \gg Y^2 \geq 1$, and this implies that $a \sim \epsilon_0$, in which case $F(h)$ can be integrated in terms of elementary functions, as shown below

$$F(h) = (h\beta)^2\left(\frac{\epsilon_s}{\epsilon_s - \epsilon_0}\right)^{3/2}\left[\frac{1/3(\gamma-1)^2 - 2(\epsilon_s - 1)(\gamma - 1) - (\epsilon_{+1} - 1)(\epsilon_{-1} - 1)}{(\gamma - 1)^{1/2}}\right]_{\epsilon_{+1}}^{U(1/h\beta)} \tag{A8}$$

The development of (A8) in a power series in $(h\beta)^{-1}$ and the inclusion of the first three terms in (A7) lead to the following approximate result for the total radiated power:

$$R_\parallel \cong \frac{Z_0}{2\epsilon_s}\left[(h\beta)^{-1} - \frac{2\pi + 1}{6\pi}\epsilon_s(h\beta) - \frac{2}{3\pi}\left(\frac{X}{Y-1}\right)^{1/2}\frac{Y-2}{Y^3}(h\beta)^2\right] \tag{A9}$$

where $X \gg Y^2 \geq 1$.

APPENDIX B: EVALUATION OF $R_\perp$

The k integration in (31) can be performed using the calculus of residues according to the following method:

1. Extend the range of the k integration along the entire real k axis, as in section 3.1.
2. Split the fourth-power sine function into a combination of exponentials as in section 3.1.
3. Make use of the identity, $J_0^2(x) = (1/\pi)\int_0^\pi J_0(2x \sin\phi)d\phi$ to replace J_0^2 in (31) and then invert the order of integration of ϕ and k.
4. Follow the method of *Mittra and Deschamps* [1963] to perform the k integration by the calculus of residues.
5. Perform the 'ϕ' integration, find the real part of the resultant expressions, and drop all terms of the order of a/h or smaller.

After the above steps are carried out, the following result is obtained for the radiation resistance of the perpendicular antenna:

$$R_\perp = C\int_0^{2\pi} d\psi \int_0^{\theta_r} d\theta \frac{n^3(n^2 - \epsilon_0)\sin^3\theta}{G(\theta)} \cdot\left[\cos^2\psi + \frac{\epsilon_d^2}{(n^2 - \epsilon_{+1})(n^2 - \epsilon_{-1})}\right]F(\psi, \theta) \tag{B1}$$

where $n = (k_-/\beta)$, $G(\theta)$ is defined in (23), $\epsilon_d = ½\,(\epsilon_{+1} - \epsilon_{-1})$, $C = Z_0(h\beta/\pi)^2/8$,

$$F(\psi, \theta) = \left(\frac{\sin \lambda n_x}{\lambda n_x}\right)^4 J_0^2[b\beta(n^2 - n_x^2)^{1/2}]$$

$n_x = n \sin\theta\cos\psi$, and $\lambda = h\beta/2$. The remaining integrations in (B1) are more easily handled after a change of variable such that

$$p = n\sin\theta\cos\psi \qquad q = n(1 - \sin^2\theta\cos^2\psi)^{1/2} \tag{B2}$$

Equation 11, along with (B2), can be used to determine the Jacobian for the transformation and the new expression for (B1) becomes

$$R_\perp = +\frac{4C}{(|\epsilon_s\epsilon_0|)^{1/2}}\int_0^\infty dp \cdot\int_{q_0}^\infty\left(p^2 + \frac{\Omega}{s-1}\right)\left(\frac{\sin\lambda p}{\lambda p}\right)^4 J_0^2(b\beta q)Q(q, p)q\,dq \tag{B3}$$

where $\lambda = h\beta/2$, $\Omega = \epsilon_d^2\,\epsilon_0/(\epsilon_0 - \epsilon_s)$,

$$Q(q, p) = \left(\frac{s - \epsilon_0}{s - a}\right)^{1/2} \cdot\left[(s - \epsilon_{+1})(s - \epsilon_{-1}) - \left(\frac{\epsilon_0 - \epsilon_s}{\epsilon_0}\right)(s - 1)p^2\right]^{-1/2}$$

$s = p^2 + q^2$, and q_0 = the unique positive real value of q, which satisfies the equation $Q^{-2}(q_0, p) = 0$. The approximation of (B3) is considerably simplified if we consider only values of the plasma parameters such that $X \gg Y^2 > 1$. In this case $a \approx \epsilon_0$, and the 'q' integration in (B3) can be easily performed in the limit as $b \to 0$ to yield

$$R_\perp = \frac{2C}{(|\epsilon_s\epsilon_0|)^{1/2}}\int_0^\infty dp\left(\frac{\sin\lambda p}{\lambda p}\right)^4 \cdot\{p^2[2\ln(2/b\beta) - \ln P(p)] + \Omega'\cos^{-1}V(p)\} \tag{B4}$$

where

$$P(p) = \left[\left(\alpha^2p^2 + \frac{2X}{Y^2 - 1}\right)^2 + 4\frac{X^2}{Y^2 - 1}\right]^{1/2}$$

$$V(p) = \left((\alpha p)^2 + \frac{2X}{Y^2 - 1}\right)P^{-1}(p)$$

$$\Omega' = X/(Y^2 - 1)^{1/2}$$

$$\alpha = \left(\frac{\epsilon_0 - \epsilon_s}{\epsilon_0}\right)^{1/2} \approx (Y^2/Y^2 - 1)^{1/2}$$

In the above integral, the term involving $\ln(2/b\beta)$ can be integrated directly. The remaining terms can be evaluated by the method used in appendix A; i.e., the integral is split into two parts as in (A4) and the sine function is replaced by its argument in the first integral, whereas in the second integral, where $p > (h\beta)^{-1}$, the functions $\ln P(p)$ and $\cos^{-1} V(p)$ are expanded in inverse powers of p, and only the first two terms are retained. The resultant integrals can all be evaluated in a straightforward fashion with the following result:

$$R_{\perp} = \frac{Z_0 \zeta^{-1}}{\pi\alpha(|\epsilon_0|)^{1/2}} \left[\ln\left(\frac{2h}{b\alpha}\right) - 1 - \frac{1}{4}\zeta^2 + \frac{(Y+1)^{3/2}}{12}\zeta^3 \right] \quad \text{(B5)}$$

where $\alpha = (Y^2/Y^2-1)^{1/2}$, and $\zeta = h\beta\epsilon_s^{1/2}/\alpha$.

The condition for the rapid convergence of the series expansion of $\ln P(p)$ in inverse powers of p (i.e., $p^2 \gg [2X/\alpha^2(Y^2-1)^{1/2}]$) sets the constraint on $(h\beta)$ for the validity of (B5):

$$(h\beta)^2 \ll \frac{\alpha^2(Y^2-1)^{1/2}}{2X} \quad \text{(B6)}$$

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