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A Comment on Ray Tracing in an Anisotropic Medium

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An improvement on numerical integration of Yabroff's ray-tracing equations is suggested wherein at each step the refractive-index vector is corrected in both magnitude and direction.

Yabroff (1961) gave a restatement of Haselgrove's equations (Haselgrove, 1955) for ray tracing in an anisotropic inhomogeneous medium. Yabroff's formulation is overdetermined, i.e., uses one more equation that Haselgrove's. Some of the differential equations are integrated to yield the spatial components of the refractive-index vector $\vec{\rho}$. This vector has the direction of the local wave normal and the magnitude of the refractive index $\mu(\psi)$, where ψ is the angle between the wave normal and the local magnetic field. Because of the overdetermination of the equations, almost any numerical integration technique will result in the magnitude of $\vec{\rho}$ being somewhat different from μ. Yabroff recognized this tendency and suggested the following technique: At each integration step, the components $\rho_{0\xi}$ of $\vec{\rho}_0$ should be corrected to $\rho_{1\xi}$ by the formula:

$$\rho_{1\xi}=\rho_{0\xi}\left(\frac{\mu}{|\vec{\rho}_0|}\right). \tag{1}$$

In this technique only the *magnitude* of the vector $\vec{\rho}$ is changed, yielding a vector $\vec{\rho}_1$. This is not too serious when the medium is approximately isotropic. The technique may not work very well when the refractive index varies rapidly with wave-normal direction. The failure of some ray tracings may result from the use of the above technique. Thus, if the angle of $\vec{\rho}$ at some point along the ray path is just slightly larger than the limiting cone of propagation, then an imaginary value of μ would be calculated, and the ray tracing would then cease.

When the refractive index becomes highly anisotropic (for example, when the direction of propagation lies close to the limiting cone), a better technique is to choose a correction vector, $\vec{\rho}_c$, which has the smallest magnitude sufficient to bring $\vec{\rho}$ to the refractive-index surface $\mu(\psi)$. One can visualize that the numerical errors cause the correct value of $\vec{\rho}$ to lie within a sphere of the calculated $\vec{\rho}$. An approximate numerical method of calculating the correction vector is to proceed as follows: (a) calculate $\rho_{1\xi}$ from equation (1); (b) calculate the angle ψ_2 for which the refractive index has the magnitude of the original $\vec{\rho}_0$; choose ψ_2 closest to ψ_0, the angle of $\vec{\rho}_0$; call the associated vector $\vec{\rho}_2$; (c) by assuming a straight line variation of μ from $\vec{\rho}_1$ to $\vec{\rho}_2$, the correction vector $\vec{\rho}_c$ can now be determined by simple geometry by dropping a perpendicular from $\vec{\rho}_0$ to that straight line. Figure 1 shows the above construction. The corrected vector is $\vec{\rho}_3$.

If $\vec{\rho}_1$ has imaginary components, then $\vec{\rho}_2$ is the desired new vector; likewise, if $\vec{\rho}_2$ has imaginary components, then $\vec{\rho}_1$ is the desired new vector.

The angle ψ_2 can be determined from the refractive-index equations. For example, using the $\mathscr{K}$ notation of Hines (1957),

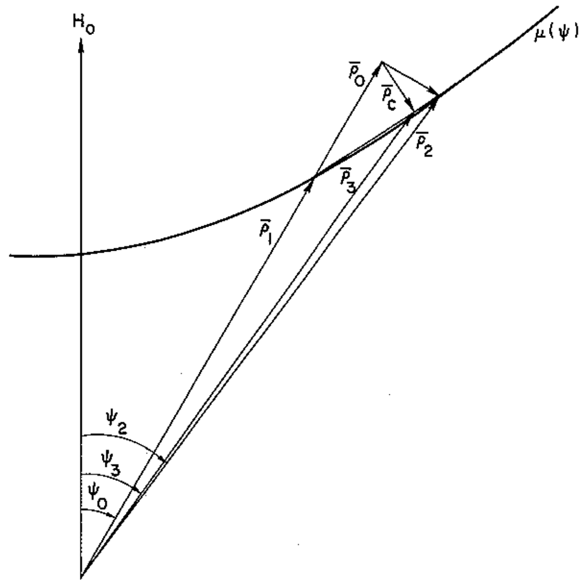

FIGURE 1. *Correction of refractive-index vector.*

$$\cos^2\psi_2 = \frac{-\mathscr{K}_2\rho_0^4 + [\mathscr{K}_1\mathscr{K}_2 + (\mathscr{K}_2^2 + \mathscr{K}_3^2)]\rho_0^2 - (\mathscr{K}_2^2 + \mathscr{K}_3^2)\mathscr{K}_1}{(\mathscr{K}_1 - \mathscr{K}_2)\rho_0^4 - [\mathscr{K}_1\mathscr{K}_2 - (\mathscr{K}_2^2 + \mathscr{K}_3^2)]\rho_0^2} . \quad (2)$$

The fourfold ambiguity in ψ_2 may be resolved by choosing the value of ψ_2 which lies in the same quadrant as ψ_0.

This new technique has been added to the ray tracing originally developed by Kimura (1966). As a further refinement, the integration step size is halved when the correction vector becomes comparable with μ. The program will now successfully integrate ray paths which formerly terminated prematurely.

The program has been found to be in good agreement by comparison with previous ray tracings, as well as the analytical ray-tracing curves calculated by Thorne and Kennel (1967).

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