

Propagation in Multicomponent Plasmas

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Abstract. A simple concise formulation of the problem of propagation in multicomponent plasmas with static magnetic fields is given. Application to plasmas, such as the ionosphere, containing electrons and multiple positive ions is considered. For each ionic species beyond the first, a multiple-ion resonance and a multiple-ion cutoff frequency are found for propagation perpendicular to the static magnetic field as well as a cutoff and the expected ion gyrofrequency resonance for the left circularly polarized (Alfvén) mode. Also, for each additional ion a crossover frequency is found for which the two longitudinally propagating modes and the transverse extraordinary mode have the same phase velocity. If a crossover frequency moves through the frequency of a wave propagating in a slowly varying medium, the polarization of the wave is changed from predominately right circular to left circular or vice versa.

Introduction. Examination of the very low frequency (VLF) recordings of the Alouette 1 satellite showed the existence of emissions at frequencies of a few kilocycles which were apparently generated in the immediate vicinity of the satellite [Brice *et al.*, 1964; Brice and Smith, 1964]. It was suggested by Brice and Smith that the lower frequency cutoff of these emissions was the lower hybrid resonance frequency f_r for the plasma surrounding the satellite. For a moderately dense plasma containing electrons and a single ionic species, this resonance frequency is given by

$$\frac{1}{Mf_r^2} = \frac{1}{f_0^2} + \frac{1}{f_H^2} \quad (1)$$

where f_0 is the electron plasma frequency, f_H the electron gyrofrequency, and M the mass-to-charge ratio of the ions relative to that of the electrons. In the ionosphere we expect more than one ionic species, so that it is of interest to determine the lower hybrid resonance frequency for a plasma containing multiple ion species. This interest has led to examination of the problem of propagation in any multicomponent plasma. Hines [1957], Buchsbaum [1960], Yakimenko [1962], and Gintsburg [1963] have considered this problem. Hines gave the refractive index for many ions. In addition, he showed that for very low frequencies in the ionosphere the upper frequency cutoff

for propagation transverse to the static magnetic field was at a frequency well above the ion gyrofrequency. This cutoff frequency is referred to as the lower hybrid resonance [Stix, 1962]. Buchsbaum showed that in the presence of two ionic species an additional resonance was found for propagation transverse to the static magnetic field at a frequency between the two ionic gyrofrequencies. Yakimenko and Gintsburg, considering propagation of Alfvén waves in the presence of multiple ions, showed that a resonance existed at each ion gyrofrequency with a cutoff at intermediate frequencies. The derivation below proceeds in a straightforward manner to yield relatively simple formulas describing propagation characteristics in a multiple component plasma. We deduce that the addition of each ionic species beyond the first introduces two additional resonances, a cutoff, and a crossover frequency. The crossover frequency, initially named for a frequency at which three limiting modes have the same phase velocity, is shown also to result in a change of modes for waves propagating in a slowly varying medium. Interesting features of propagation arising from the presence of multiple ion species are illustrated graphically.

Refractive indices for the principal modes. In the derivation below, we consider small signal sinusoidal plane waves in a uniform cold plasma containing a static magnetic field. For completeness, the derivation is started from Maxwell's equations and the Lorentz force law. From Maxwell's equations

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$$\nabla \times \mathbf{H} = \mathbf{J} + \epsilon_0 \partial \mathbf{E} / \partial t \quad (2)$$

$$= \mathbf{J} + i\omega \epsilon_0 \mathbf{E} \quad (3)$$

$$= i\omega \mu^2 \epsilon_0 \mathbf{E} \quad (4)$$

where μ is the refractive index, and μ^2 the effective relative dielectric constant for the medium. Thus μ^2 can be simply related to the current \mathbf{J} by

$$\mathbf{J} = i\omega \epsilon_0 (\mu^2 - 1) \mathbf{E} \quad (5)$$

If the plasma contains many different species of charged particles, the total current is, of course, the sum of the currents arising from each of the species. In a rectangular coordinate system with a static magnetic field of magnitude B/μ_0 in the z direction, the velocities for a given particle species with charge q_r and mass m_r are found from the Lorentz force law

$$i\omega m_r v_x = q_r (E_x + v_y B) \quad (6)$$

$$i\omega m_r v_y = q_r (E_y - v_x B) \quad (7)$$

$$i\omega m_r v_z = q_r E_z \quad (8)$$

The effect of collisions can be readily included in the above equations if we assume a collisional damping term proportional to velocity. Then the left-hand side of (6) becomes

$$m_r (i\omega v_x + \nu_r v_x) = i\omega m_r v_x (1 - i\nu_r/\omega) \quad (9)$$

$$= i\omega m_r v_x (1 - iZ_r) \quad (10)$$

and similarly for the other two equations. The effect of collisions can be included by replacing the mass of the particle m_r by $m_r(1 - iZ_r)$.

The equations are further simplified by the introduction of polarized coordinates $(x + iy)/\sqrt{2}$, $(x - iy)/\sqrt{2}$, z , and by using the convenient subscript notation of *Buneman* [1961], so that, for example,

$$v_1 = (v_x + iv_y)/\sqrt{2} \quad (11)$$

$$v_{-1} = (v_x - iv_y)/\sqrt{2} \quad (12)$$

$$v_0 = v_z \quad (13)$$

From (6), (7), and (8) we obtain

$$i\omega m_r v_1 = q_r (E_1 - i\nu_1 B) \quad (14)$$

$$i\omega m_r v_{-1} = q_r (E_{-1} + i\nu_{-1} B) \quad (15)$$

$$i\omega m_r v_0 = q_r E_0 \quad (16)$$

These equations can be collectively written as

$$i\omega v_p \left(1 + \frac{pq_r B}{m_r \omega} \right) = \frac{q_r}{m_r} E_p \quad (17)$$

where $p = (-1, 0, +1)$. The current J_{pr} due to the r th species with number density N_r is

$$J_{pr} = \frac{N_r q_r^2 E_p}{i\omega m_r [1 + (pq_r B/m_r \omega)]} \quad (18)$$

$$= -i\epsilon_0 \omega \frac{X_r}{1 + pY_r} E_p \quad (19)$$

where X_r is the ratio of the square of the plasma frequency of the r th species to the square of wave frequency, and Y_r is the ratio of the gyrofrequency of particles of the r th species to the wave frequency, with the sign of Y_r being the same as the sign of the charge of the r th species. The total current is then the sum of the currents due to each species, so that we obtain, using (5),

$$\mu_p^2 = 1 - \sum_r \frac{X_r}{pY_r + 1} \quad (20)$$

Equation 20 gives the refractive indices for the three principal modes of propagation in the plasma, the right (+1) and the left (-1) circularly polarized longitudinal modes and the transverse plasma mode (0). The terms longitudinal and transverse refer here to the direction of propagation with respect to the static magnetic field.

Refractive indices for arbitrary directions of propagation. The derivation below follows class notes prepared by *Buneman* (private communication).

Without loss of generality, we can assume that the propagation vector \mathbf{k} lies in the x - z plane, making an angle θ with the z axis. The wave equation can be written

$$c^2 \nabla \times \nabla \times \mathbf{E} = -\epsilon^{(r)} \partial^2 \mathbf{E} / \partial t^2 \quad (21)$$

so that

$$-c^2 [k^2 \mathbf{E} - \mathbf{k}(\mathbf{k} \cdot \mathbf{E})] = \epsilon^{(r)} \omega^2 \mathbf{E} \quad (22)$$

The relative dielectric constant $\epsilon^{(r)}$ in these equations must be interpreted as a tensor. This complication is obviated by the use of polarized coordinates. The polarized components of (22) are given by

$$c^2 k^2 E_p - c^2 k_p (\mathbf{k} \cdot \mathbf{E}) = \omega^2 \mu_p^2 E_p \quad (23)$$

($p = -1, 0, +1$)

From this equation we obtain

$$E_p = c^2 [\mathbf{k} \cdot \mathbf{E} / (c^2 k^2 - \omega^2 \mu_p^2)] k_p \quad (24)$$

Letting $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$ and W (the phase velocity) = $\omega/|k|$, (24) becomes

$$E_p = \frac{\hat{\mathbf{k}} \cdot \mathbf{E}}{1 - W^2/W_p^2} \hat{k}_p \quad (25)$$

It is seen that, when the divergence of E is zero ($\mathbf{k} \cdot \mathbf{E} = 0$), nonzero values of electric field can be obtained only for the three principal modes ($W = W_p$). For this reason these modes can also be called the 'charge-free' modes.

Let us now consider the summation

$$\sum_p \hat{k}_{-p} E_p = \hat{k}_{-1} E_1 + \hat{k}_1 E_{-1} + \hat{k}_0 E_0 \quad (26)$$

$$= \hat{k}_x E_x + \hat{k}_y E_y + \hat{k}_z E_z \quad (27)$$

$$= \hat{\mathbf{k}} \cdot \mathbf{E} \quad (28)$$

Multiplying both sides of equation 25 by \hat{k}_{-p} and summing over p , we find

$$\sum_p \hat{k}_{-p} E_p = (\hat{\mathbf{k}} \cdot \mathbf{E}) \sum_p \frac{\hat{k}_p \hat{k}_{-p}}{1 - W^2/W_p^2} = \hat{\mathbf{k}} \cdot \mathbf{E} \quad (29)$$

Assuming that $\hat{\mathbf{k}} \cdot \mathbf{E}$ is nonzero, we obtain

$$\sum_p \frac{\hat{k}_p \hat{k}_{-p}}{1 - W^2/W_p^2} = 1 \quad (30)$$

For the orientation of the axes chosen,

$$\hat{k}_x = \sin \theta \quad \hat{k}_y = 0 \quad \hat{k}_z = \cos \theta \quad (31)$$

so that

$$\hat{k}_1 = \hat{k}_{-1} = \sin \theta / \sqrt{2} \quad \hat{k}_0 = \cos \theta \quad (32)$$

Equation 30 then becomes

$$\frac{1}{2} \sin^2 \theta \left[\frac{1}{1 - W^2/W_1^2} + \frac{1}{1 - W^2/W_{-1}^2} \right] + \frac{\cos^2 \theta}{1 - W^2/W_0^2} = 1 \quad (33)$$

which can be rewritten as

$$\frac{1}{2} \sin^2 \theta \left[\frac{W_1^2}{W_1^2 - W^2} + \frac{W_{-1}^2}{W_{-1}^2 - W^2} - 2 \right] + \cos^2 \theta \left[\frac{W_0^2}{W_0^2 - W^2} - 1 \right] = 0 \quad (34)$$

or

$$\frac{1}{2} \sin^2 \theta \left[\frac{W^2 W_1^2 + W^2 W_{-1}^2 - 2W^4}{(W_1^2 - W^2)(W_{-1}^2 - W^2)} \right] + \cos^2 \theta \left[\frac{W^2}{W_0^2 - W^2} \right] = 0 \quad (35)$$

Introducing the quantity

$$W_*^2 = \frac{1}{2}(W_1^2 + W_{-1}^2) \quad (36)$$

we obtain

$$(W^2 - W_*^2)(W^2 - W_0^2) \sin^2 \theta + (W^2 - W_1^2)(W^2 - W_{-1}^2) \cos^2 \theta = 0 \quad (37)$$

It is seen that for zero angle of propagation the two modes are the +1 and the -1 principal modes. For ninety degrees, we obtain the 0 principal mode and the transverse extraordinary (e) mode. For the e mode, the square of the phase velocity is the average of the squares of the phase velocities of the two longitudinal (+1 and -1) modes. These four modes obtained for the limiting angles of 0° and 90° will be referred to as the limiting modes.

For any arbitrary direction of propagation the phase velocities and hence refractive indices of the two characteristic modes can be obtained from (37) in terms of the phase velocities of the four limiting modes.

Polarization. For the +1 principal mode, the polarization is right circular; for the -1 mode, left circular; and for the 0 mode, linear. Further, it is seen from (25) that for these principal modes there is no component of electric field in the wave normal direction. For $W \neq W_p$, the ratios of the $r - f$ electric fields are obtained from (25) as

$$E_x : iE_y : E_z = \frac{\sin \theta}{2} \left[\frac{W_1^2}{W_1^2 - W^2} + \frac{W_{-1}^2}{W_{-1}^2 - W^2} \right] : \frac{\sin \theta}{2} \left[\frac{W_1^2}{W_1^2 - W^2} - \frac{W_{-1}^2}{W_{-1}^2 - W^2} \right] : \cos \theta \left[\frac{W_0^2}{W_0^2 - W^2} \right] \quad (38)$$

The four values of interest are the components E_x and E_y together with E_n , the component in the wave normal direction, and E_w , the compo-

ment in the wave front and the $x-z$ plane, where

$$E_n = E_x \cos \theta + E_z \sin \theta \quad (39)$$

$$E_w = E_x \cos \theta - E_z \sin \theta \quad (40)$$

In discussing the magnitudes of the field components of the wave, it is convenient to use as a standard the fields obtained with the same Poynting flux for propagation in free space. For linear or circular polarization, the ratio of wave electric field in the wave front, E_w , to the equivalent free space value, E_{fs} , is given by

$$E_w/E_{fs} = \mu^{-1/2} \quad (41)$$

while for the magnetic field of the wave,

$$\frac{H_w}{H_{fs}} = \mu^{1/2} \quad (42)$$

Group refractive indices. To obtain the group refractive indices, we examine initially the principal modes, then the transverse extraordinary mode, and finally the modes for arbitrary angles. The group refractive index μ_g can be obtained from

$$\mu\mu_g = \mu^2 + \frac{\omega}{2} \frac{\partial \mu^2}{\partial \omega} \quad (43)$$

Using (20), we obtain

$$\mu_p\mu_{gp} = 1 - \frac{1}{2} \sum_r \frac{p Y_r X_r}{(p Y_r + 1)^2} \quad (44)$$

For the transverse extraordinary mode, we obtain from (35)

$$2\mu_e^{-2} = \mu_1^{-2} + \mu_{-1}^{-2} \quad (45)$$

Differentiating (45) with respect to ω and multiplying by $\omega/2$, we obtain

$$\frac{2}{\mu_e^4} \frac{\omega}{2} \frac{\partial \mu_e^{-2}}{\partial \omega} = \frac{1}{\mu_1^4} \frac{\omega}{2} \frac{\partial \mu_1^{-2}}{\partial \omega} + \frac{1}{\mu_{-1}^4} \frac{\omega}{2} \frac{\partial \mu_{-1}^{-2}}{\partial \omega} \quad (46)$$

Adding the left- and right-hand sides of (45) and (46), and using (36), we get

$$\frac{2}{\mu_e^4} (\mu_e\mu_{ge}) = \frac{1}{\mu_1^4} (\mu_1\mu_{g1}) + \frac{1}{\mu_{-1}^4} (\mu_{-1}\mu_{g-1}) \quad (47)$$

Examination of (44) and (47) shows that the product of the refractive index and group refractive index is very simply computed for the limiting modes. Since (20) gives the value of μ^2

rather than μ , factors of μ in (47) have deliberately not been cancelled.

The formula for calculating the group refractive index for arbitrary directions of propagation is obtained by differentiating (37) with respect to ω and multiplying by $\omega/2$, as before, and then adding twice the left- and right-hand sides of (37). This process yields the somewhat lengthy but reasonably symmetrical form

$$\begin{aligned} \sin^2 \theta & \left[\left(\frac{\mu\mu_g}{\mu^4} - \frac{\mu_0\mu_{g0}}{\mu_0^4} \right) (\mu^{-2} - \mu_e^{-2}) \right. \\ & \left. + \left(\frac{\mu\mu_g}{\mu^4} - \frac{\mu_e\mu_{ge}}{\mu_e^4} \right) (\mu^{-2} - \mu_0^{-2}) \right] \\ & + \cos^2 \theta \left[\left(\frac{\mu\mu_g}{\mu^4} - \frac{\mu_1\mu_{g1}}{\mu_1^4} \right) (\mu^{-2} - \mu_{-1}^{-2}) \right. \\ & \left. + \left(\frac{\mu\mu_g}{\mu^4} - \frac{\mu_{-1}\mu_{g-1}}{\mu_{-1}^4} \right) (\mu^{-2} - \mu_{-1}^{-2}) \right] = 0 \quad (48) \end{aligned}$$

Resonances and cutoff frequencies. In what follows, the term 'resonance' will be used to describe frequencies at which the refractive index is infinite (zero phase velocity) and the term 'cutoff' for frequencies at which the refractive index is zero (infinite phase velocity). From (20) it is apparent that for the principal modes the resonance frequencies are those for which

$$p Y_r + 1 = 0 \quad (49)$$

remembering that Y_r is positive for positively charged species and negative for negatively charged species. Thus for left (-1) circularly polarized waves, a resonance is found at the gyrofrequency of each of the positively charged constituents; for right ($+1$) circularly polarized waves the resonances occur at the gyrofrequencies of the negatively charged constituents. It is readily apparent that for the plasma mode ($p = 0$) there are no resonances.

From (36) it is seen that resonances occur for the transverse extraordinary modes (e) when the squares of the refractive indices of the $+1$ and -1 longitudinal modes are equal in magnitude and opposite in sign. Using (20), we obtain this resonance from

$$1 - \sum_r \frac{X_r}{1 + Y_r} + 1 - \sum_r \frac{X_r}{1 - Y_r} = 0 \quad (50)$$

$$\sum_r \left(\frac{X_r}{1 + Y_r} + \frac{X_r}{1 - Y_r} \right) = 2 \quad (51)$$

$$\sum_r \frac{X_r}{1 - Y_r^2} = 1 \tag{52}$$

For arbitrary angles we note that (37) is of the form

$$\alpha_4 W^4 + \alpha_2 W^2 + \alpha_0 = 0 \tag{53}$$

For a resonance, we require $\alpha_0 = 0$. This condition is readily found to be

$$\left[1 - \sum_r \frac{X_r}{1 - Y_r^2} \right] \sin^2 \theta + \left[1 - \sum_r X_r \right] \cos^2 \theta = 0 \tag{54}$$

if $\sum X_r$ is not equal to 1.

For cutoff frequencies we require zero refractive index, so that for the principal modes

$$\sum_r \frac{X_r}{p Y_r + 1} = 1 \tag{55}$$

From (36) it is apparent that a cutoff frequency (infinite phase velocity) for either the +1 or -1 modes is also a cutoff for the transverse extraordinary mode. Further, from consideration of (37) it can be seen that a cutoff frequency for any of the limiting modes will also be a cutoff for one of the characteristic modes for all angles.

It is of interest to consider the polarization of waves at frequencies near cutoff and resonance frequencies. It is seen from (41) and (42) that cutoffs are characterized by large electric fields in the wave front. Further, of the four limiting modes, only the transverse extraordinary modes can have a component of wave electric field in the direction of propagation. For this mode it can be shown from (38) and (39) that

$$E_x = 0 \tag{56}$$

$$\frac{E_x}{iE_y} = \frac{E_n}{iE_y} = \frac{W_1^2 - W_{-1}^2}{W_1^2 + W_{-1}^2} \tag{57}$$

At a cutoff for the transverse extraordinary mode, either W_1^2 or W_{-1}^2 tends toward infinity, so that the magnitudes of E_x and E_y become equal. Thus the electric field of the wave is circularly polarized and consists of one component in the wave normal direction, while the other lies in the wave front and is perpendicular to the static magnetic field. The magnetic field of the wave is parallel to the static magnetic field.

For resonance frequencies, the component of

electric field in the wave front becomes small, so that the principal modes are dominated by large magnetic fields of the wave. In these cases, the resonances are termed 'electromagnetic.' However, for the transverse extraordinary mode near a resonance frequency,

$$W_1^2 \cong -W_{-1}^2 \tag{58}$$

$$W_e^2 \cong 0 \tag{59}$$

and, using (36) and (57), we see that

$$\frac{E_n}{iE_y} \cong \frac{W_1^2}{W_e^2} = \frac{\mu_e^2}{\mu_1^2} \tag{60}$$

Thus for the *e* mode the dominant wave field near resonance is the large electric field component in the direction of propagation. Resonances for this case are termed 'electrostatic.'

Propagation with multiple ion species. As an illustration of the formulas derived above, their application to a plasma containing electrons and multiple positive ionic species is considered. For frequencies of the order of or greater than the electron gyrofrequency, ion effects on propagation are extremely small. We will therefore concentrate our attention on frequencies which are considerably less than the electron gyrofrequency. For these frequencies there are no resonances or cutoffs for the right circularly polarized (+1) mode if we exclude negative ions from consideration. Furthermore, unless the electron plasma frequency is much less than the electron gyrofrequency, the plasma (0) mode will be a nonpropagating mode for the frequencies of interest. It is shown below that for medium and high density plasmas most of the frequencies of interest are independent of the electron plasma frequency, so that in Figures 1 through 4 the frequency has been normalized to the electron gyrofrequency, the ratio being denoted by λ so that

$$\lambda = f/f_H = 1/Y_e \tag{61}$$

To illustrate propagation characteristics, the refractive index or phase velocity, or the squares of these quantities, can be used. Because of the relationship given by (36), we have chosen, for the most part, to plot phase velocity squared as a function of λ , for the +1 or right circularly polarized wave (*R*), for the -1 or left circularly polarized wave (*L*), and for the transverse extraordinary mode (*e*). For all these figures a ratio of electron gyrofrequency to electron

plasma frequency of 0.4 was used. All ions were assumed singly charged.

Figures 1a, b, and c show the phase velocity squared as a function of λ for plasmas containing only one ion (hydrogen), two ions (75% H⁺, 25% He⁺), and three ions (75% H⁺, 20% He⁺, 5% O⁺), respectively.

For one ion (Figure 1a), the two frequencies of interest are the well-known ion gyrofrequency resonance for the *L* mode and the lower hybrid resonance for the *e* mode, the former resonance being electromagnetic and the latter electrostatic. The two-ion case shows many new features: a cutoff for the *e* and *L* modes, and an additional resonance for the *e* mode and one for the *L* mode, and a frequency at which the phase velocities for the *R*, *L*, and *e* modes are all equal. The additional resonance frequency for the *L* mode is, of course, the gyrofrequency of the second ion. For reasons that will become apparent later, the term 'lower hybrid resonance' will be used for the *e* mode resonance frequency which is above the highest ion gyrofrequency and below the electron gyrofrequency, i.e., the highest *e* mode resonance shown in Figure 1. The existence of an additional resonance for the *e* mode, as seen in Figure 1b, was noted by *Buchsbaum* [1960]. This lower frequency *e* mode resonance will be referred to as the 'two-ion resonance.'

Similarly, the cutoff shown in Figure 1b will be called the 'two-ion cutoff.' The frequency at which the *R*, *L*, and *e* modes have the same phase velocity will be called the 'crossover frequency.'

The new features found when a second ion is added are repeated with the addition of a third ion (Figure 1c), and the general pattern obtained for four or more ions is readily apparent. For more than two ions, the additional *e* mode resonances can be referred to as 'multiple-ion resonances,' and the cutoffs as 'multiple-ion cutoffs.'

For the *R* mode it is apparent that the introduction of negative ions will introduce features similar to those shown for the *L* mode in Figures 1b and c.

Phase velocity surfaces. For consideration of propagation at arbitrary wave normal angles, it is convenient to use phase velocity surfaces, i.e., polar plots of the phase velocity as a function of the angle from the static magnetic field to

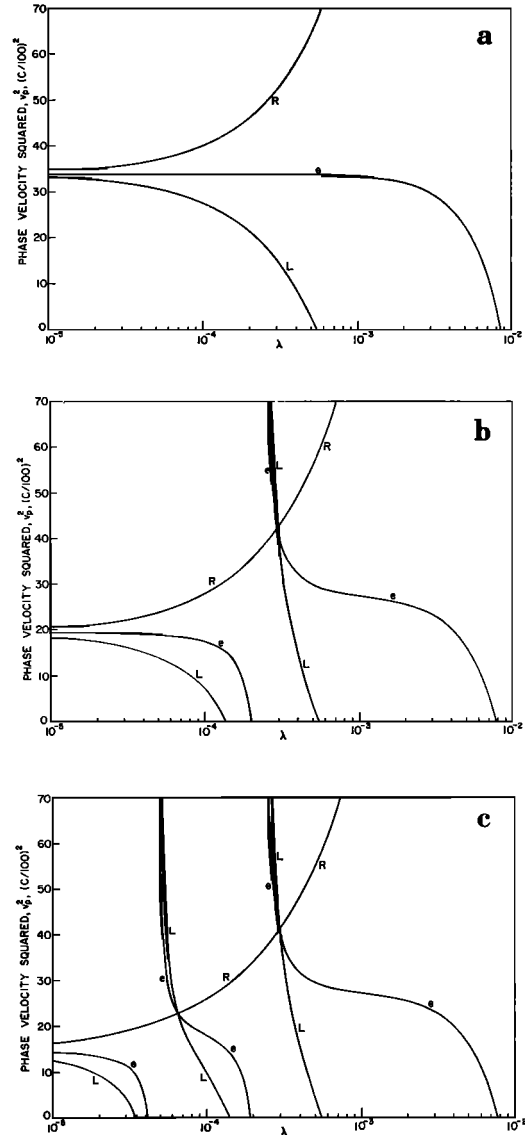


Fig. 1. The square of phase velocity for three limiting modes as a function of normalized frequency $\lambda = f/f_H$ for a plasma consisting of electrons and (a) hydrogen, (b) 75% hydrogen and 25% helium, (c) 75% hydrogen, 20% helium, and 5% oxygen.

the wave normal direction. The phase velocity can be obtained from (37), which can also be written in the form

$$2W^2 = W_0^2 \sin^2 \theta + W_e^2 (1 + \cos^2 \theta) \pm [(W_0^2 - W_e^2)^2 \sin^4 \theta + (W_1^2 - W_{-1}^2)^2 \cos^2 \theta]^{1/2} \quad (62)$$

The term contained within the radical sign of (62) is positive definite and is zero between 0° and 90° only if all four limiting modes have the same phase velocity. Neglecting this circumstance, the radical is zero only for $\theta = 0^\circ$ and $W_1 = W_{-1}$, or $\theta = 90^\circ$ and $W_0 = W_e$. For any angle other than possible 0° or 90° , one mode always has greater phase velocity than the other, the former being called the fast mode and the latter the slow mode. It is apparent, then, that as the direction of propagation is varied the fast mode for longitudinal propagation is always transformed to the fast mode for transverse propagation, and similarly for the slow mode. From the above discussion, the general nature of the phase velocity surfaces can be deduced from the phase velocities of the four limiting modes.

In Figure 1 the square of the velocity of the 0 mode is not shown, but it is small and negative. Thus for the frequencies and parameters chosen, a maximum number of three of the limiting modes can propagate. From Figure 1, it is seen that when multiple ions are introduced the fast mode in the longitudinal direction can be either the right or left circularly polarized modes.

Figures 2a and b show phase velocity surfaces for frequencies slightly below and slightly above a crossover frequency where three of the limiting modes propagate. The fast mode for longitudinal propagation is left circularly polarized in Figure 2a and right circularly polarized in Figure 2b. Figures 2c and 2d show phase velocity surfaces for frequencies at which two and one limiting modes, respectively, can propagate.

The crossover frequency. From examination of Figures 2a and b it is apparent that at the crossover frequency propagation is isotropic for the fast mode and the phase velocity follows approximately a cosine law for the slow mode. These two features are found for Alfvén propagation when the frequency tends toward zero. Experiments on Alfvén propagation are usually difficult to perform, since collisions must be exceedingly infrequent. However, this difficulty can be overcome by using plasmas containing two or more ion species and propagating at a crossover frequency.

Also, from Figures 2a and b we can anticipate that if the crossover frequency moves through the wave frequency the wave polarization will

change from predominately right circular to predominately left circular. This point is also illustrated in Figure 3, where the phase velocity squared is plotted as a function of λ ($=f/f_H$) for several wave normal angles for a plasma containing 80% hydrogen ions and 20% oxygen ions. It is seen that for all angles the fast mode passes through the crossover point and that the slope of these curves is the same for all angles other than zero. We can deduce that the group velocity for the fast mode for nonzero angle will be independent of angle. From the formulation of polarization given above it can be deduced that both the fast and slow waves are linearly polarized at the crossover frequency for all wave normal angles other than zero.

Useful approximations for ionospheric applications. In general we can assume that the ionic mass is much greater than the electron mass ($M \gg 1$). Then if we define dense plasmas as those for which $f_o^2 \gg f_H^2$, and moderately dense plasmas as those for which $Mf_o^2 \gg f_H^2$, we can say that the ionosphere is at least moderately dense at all heights above about 80 km. Frequencies of interest here are always less than the geometric mean of electron and ion gyrofrequencies, so that $f_H^2 \gg f^2$, and $f_o^2 \gg ff_H$. The usefulness of these approximations will be seen in the next section.

Propagation with two-ion species. As was discussed above, no essentially new features are added when a third ion species is introduced, so that it is instructive to derive more specific formulas for the frequencies of interest for a moderately dense plasma containing electrons (e) and two ion species (i1 and i2). For this case, the lower hybrid resonance frequency is found from (52):

$$1 + \frac{X_e}{Y_e^2 - 1} + \frac{X_{i1}}{Y_{i1}^2 - 1} + \frac{X_{i2}}{Y_{i2}^2 - 1} = 0 \quad (63)$$

Since the lower hybrid resonance (for one ion species) is of the order of the ion plasma frequency or the geometric mean gyrofrequency, whichever is less, we can assume for a plasma at least moderately dense that

$$Y_e^2 \gg 1 \quad Y_i^2 \ll 1 \quad (64)$$

and (63) becomes

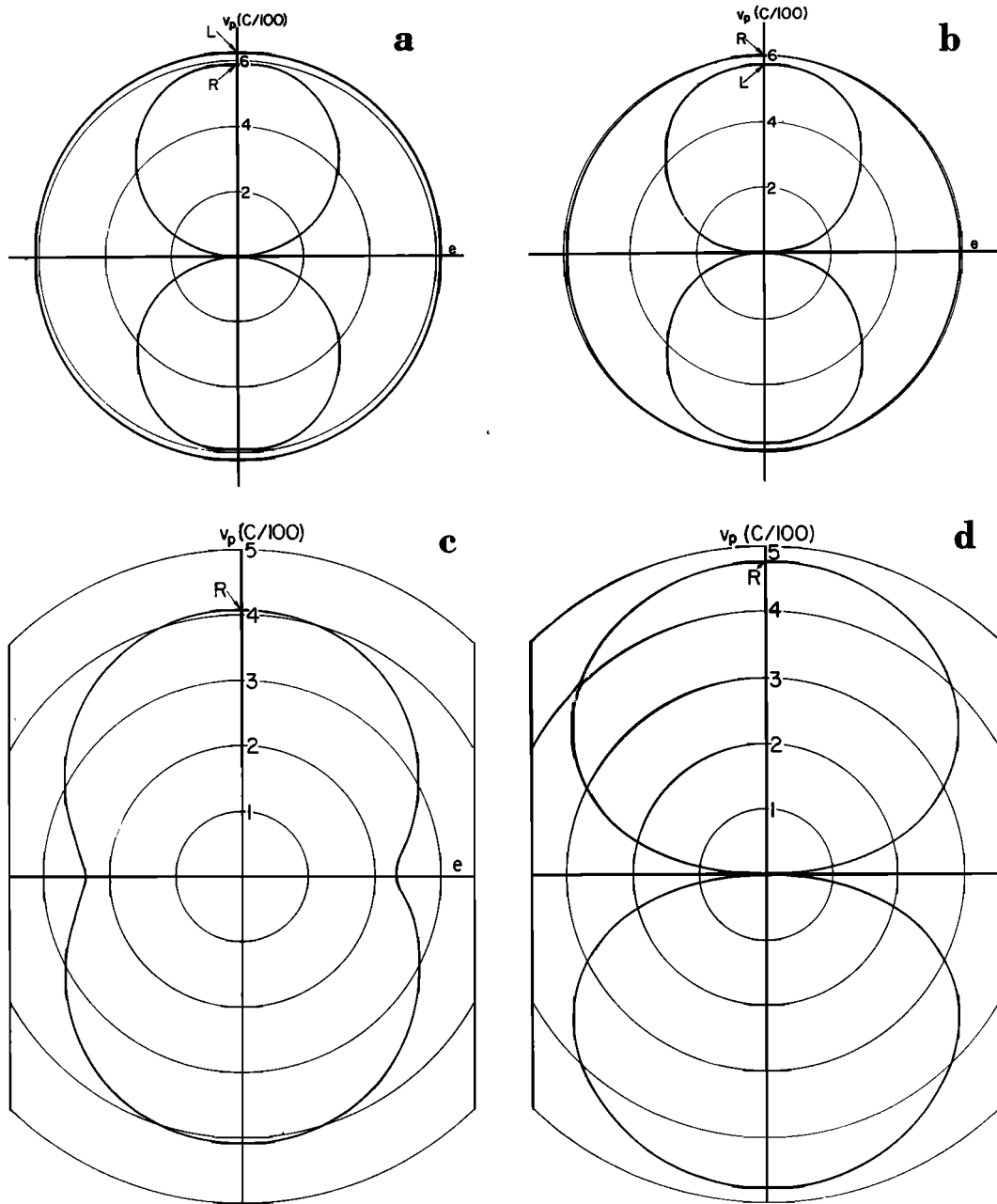


Fig. 2. Phase velocity as a function of wave normal angle for frequencies (a) slightly below a crossover, (b) slightly above a crossover, (c) at which two limiting modes propagate, and (d) at which one limiting mode propagates.

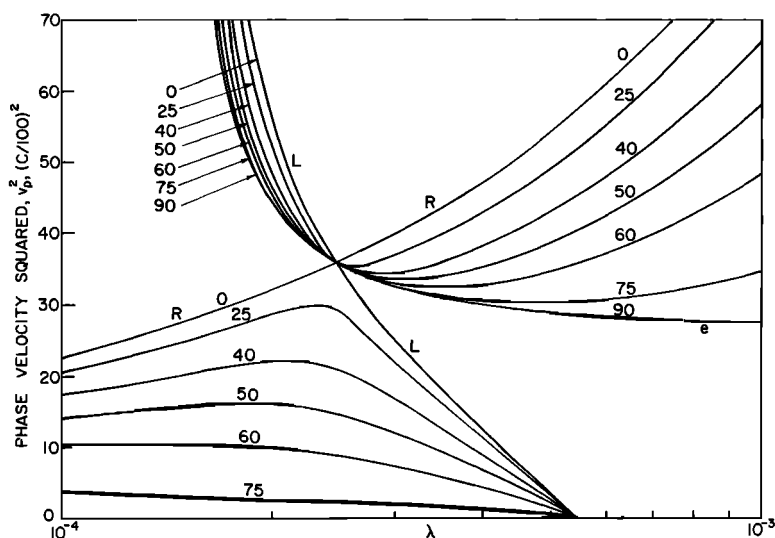


Fig. 3. The square of phase velocity as a function of normalized frequency $\lambda = f/f_H$ in the vicinity of a crossover for wave normal angles between 0° and 90° .

$$Y_e^2 + X_e = (X_{i1} + X_{i2}) Y_e^2 \quad (65)$$

If we define A_j as the fraction of the positive ion charge density occupied by the j th ion species and M_j as the mass to charge ratio of the j th ion species relative to that of the electron, then

$$X_{i,j} = X_e \frac{A_j}{M_j} \quad (66)$$

Equation 65 can then be written

$$\frac{1}{f_r^2} \sum_i \frac{A_i}{M_i} = \frac{1}{f_0^2} + \frac{1}{f_H^2} \quad (67)$$

where f_r is the lower hybrid resonance frequency. Though (67) was derived for two ions, it is equally valid for any number of ionic species.

The two-ion resonance frequency is always intermediate between the two ionic gyrofrequencies, as is the two-ion cutoff frequency and the crossover frequency. To determine the two-ion resonance frequency, equation 63 is written

$$1 + \frac{X_e}{Y_e^2 - 1} + \frac{A_1 M_1 X_e}{Y_e^2 - M_1^2} + \frac{A_2 M_2 X_e}{Y_e^2 - M_2^2} = 0 \quad (68)$$

for frequencies of the order of the ion gyrofrequencies

$$Y_e^2 \sim M^2 \quad (69)$$

so that the last two terms of (68) are of the order of MX_e/Y_e^2 , which is much greater than X_e/Y_e^2 , and for a dense or moderately dense plasma is much greater than unity. When we use these approximations, (68) becomes

$$\frac{A_1 M_1}{Y_e^2 - M_1^2} + \frac{A_2 M_2}{Y_e^2 - M_2^2} = 0 \quad (70)$$

from which we obtain the two-ion resonance frequency

$$f_{2Ir} = \frac{f_H^2}{M_1 M_2} \left(\frac{A_1 M_1 + A_2 M_2}{A_2 M_1 + A_1 M_2} \right) \quad (71)$$

The two-ion cutoff frequency f_{2rc} is obtained from (55) by putting p equal to -1 . We then obtain for two-ion species

$$\frac{X_e}{Y_e + 1} - \frac{A_1 X_e}{Y_e - M_1} - \frac{A_2 X_e}{Y_e - M_2} = 1 \quad (72)$$

and, for a plasma at least moderately dense, we obtain

$$\frac{1}{Y_e} = \frac{A_1}{Y_e - M_1} + \frac{A_2}{Y_e - M_2} \quad (73)$$

Noting that

$$\sum_i A_i = 1 \quad (74)$$

we obtain for the two-ion cutoff frequency

$$f_{2Ic} = f_H \left(\frac{A_1}{M_2} + \frac{A_2}{M_1} \right) \quad (75)$$

The crossover frequency f_{cr} is obtained from

$$\frac{X_e Y_e}{Y_e^2 - 1} - \frac{X_{i1} Y_{i1}}{Y_{i1}^2 - 1} - \frac{X_{i2} Y_{i2}}{Y_{i2}^2 - 1} = 0 \quad (76)$$

from which we find (for $Y_e^2 \gg 1$)

$$\frac{1}{Y_e^2} = \frac{A_1}{Y_e^2 - M_1^2} + \frac{A_2}{Y_e^2 - M_2^2} \quad (77)$$

so that

$$f_{cr}^2 = f_H^2 \left(\frac{A_2}{M_1^2} + \frac{A_1}{M_2^2} \right) \quad (78)$$

It should be noted that the lower hybrid resonance frequency is independent of electron density only for dense plasmas. The multiple-ion cutoff and resonance frequencies are independent of density for plasmas which are at least moderately dense, whereas the crossover frequencies are completely independent of density.

As an example of the application of the formulas derived above for the two-ion resonance, cutoff, and crossover frequencies, consider a plasma with 20% oxygen atoms and 80% protons, with an electron plasma frequency of 400 kc/s and an electron gyrofrequency of 1 Mc/s.

Then the lower hybrid resonance frequency obtained from (67) is 7.83 kc/s, and the two-ion resonance frequency is found from (71) to be 75.8 cps. The two-ion cutoff frequency of 136.5 cps is obtained from (75) and the crossover frequency of 246 cps from (78).

A number of the features of propagation in a multicomponent plasma are summarized in Figure 4. Figures 4a and b show the group velocity and phase velocity as a function of normalized frequency. In Figure 4c are sketched phase velocity surfaces (not to scale) for frequencies indicated. There are a sufficient number of surfaces shown to indicate the general nature of propagation in the frequency range where ions are important. The ion constituents are assumed to be 80% hydrogen and 20% oxygen. The ion gyrofrequencies are at a normalized frequency of about 3.41×10^{-6} and 5.46×10^{-4} . The two-ion resonance occurs at 7.83×10^{-8} . The two-ion cutoff occurs at 1.36×10^{-4} and crossover at 2.46×10^{-4} . The figure also illustrates the well-

known fact that the group velocity is zero whenever the phase velocity is either zero or infinite.

Determination of constituents. It has been shown above that the number of multiple-ion resonance frequencies, the number of multiple-ion cutoff frequencies, and the number of crossover frequencies are each one less than the number of ionic species. Also, for plasmas at least moderately dense, all these frequencies are independent of density and are functions only of the electron gyrofrequency and the masses and relative densities of the ionic constituents.

Thus, given the ion masses (or, more accurately, mass-to-charge ratios) the relative densities (A_i) of each of the ionic constituents of a moderately dense multiple-ion plasma can be determined from a knowledge of *either* all the multiple-ion resonance frequencies *or* all the multiple-ion cutoff frequencies, or all the crossover frequencies (remembering that $\sum A_i = 1$). If mass-to-charge ratios of the ions are not known, they can be found from a knowledge of the ion gyrofrequency resonances.

It is apparent that measurement of multiple-ion resonance and cutoff frequencies may provide a useful diagnostic tool for plasmas containing multiple ion species, such as the earth's ionosphere.

Related experimental data. As was noted above, VLF emissions which are believed to arise from the lower hybrid resonance have been observed in satellites [Brice and Smith, 1964]. Very strong signals, apparently associated with the proton gyrofrequency, have also been observed [Smith *et al.*, 1964]. In addition, propagation phenomena believed to arise from the lower hybrid resonance, a multiple ion cutoff, and a crossover frequency have been found. These will be discussed in detail at a later date.

Discussion. The results derived above have already been found to be extremely useful in the interpretation of rocket and satellite VLF recordings. They also have obvious potential application to the problem of determining the ionic constituents of the ionosphere by, for example, measuring the impedance of an antenna in the ionosphere as a function of frequency. Consideration of ionospheric parameters indicates a desirable frequency range for such an instrument of 10 cps to 30 kc/s. With this extended low frequency range, the effects associated with the lower frequency multiple ion cut-

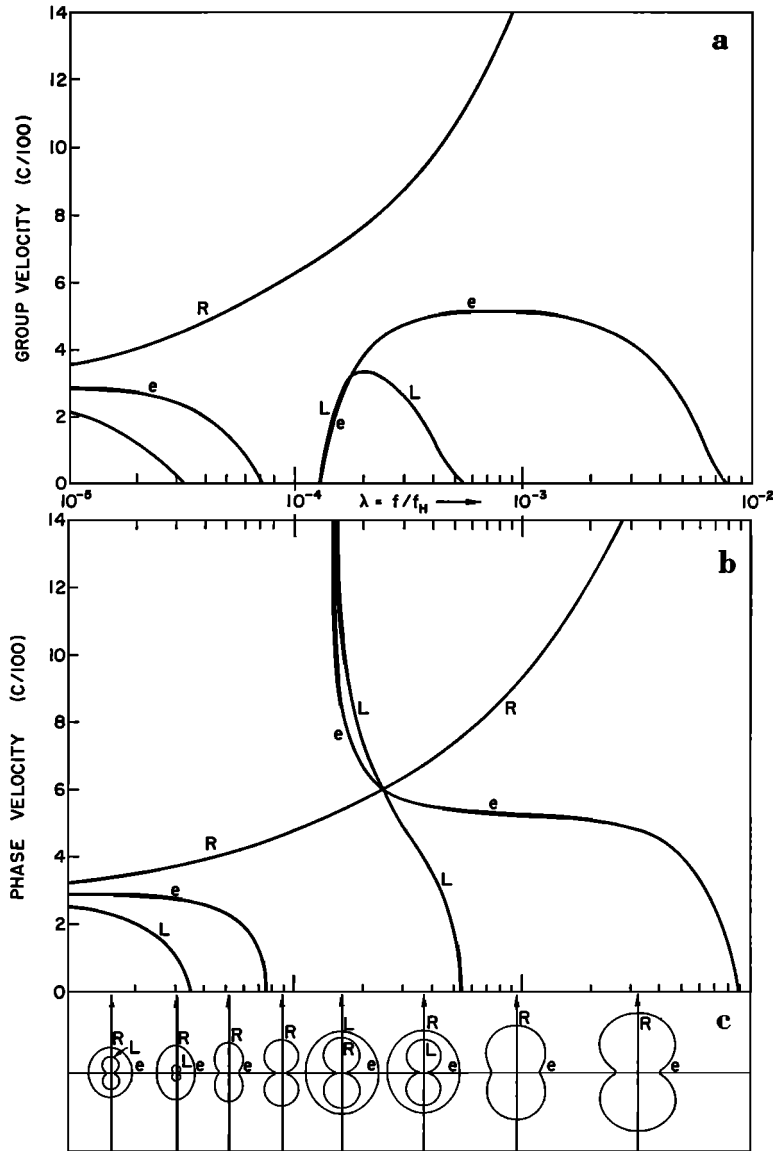


Fig. 4. (a) Group velocity, (b) phase velocity, and (c) phase velocity surfaces as a function of normalized frequency $\lambda = f/f_H$.

offs and resonances may well be observed. Simultaneous measurements of electron plasma and gyrofrequency would be of interest.

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