first paper,[1] Unz writes Maxwell's equations for a coordinate system fixed in space. The plasma has a mean flow velocity v_0 with respect to this coordinate system. Hence, Unz's (9) for the motion of an electron is properly written if one interprets d/dt as a material derivative [*i.e.*, $d/dt=(\partial/\partial t + v_T\cdot\nabla)$]. However, in carrying out the analysis Unz treated d/dt as $\partial/\partial t$. The present author has reworked the analysis of Unz using the proper form for d/dt. The correct dispersion relation may be obtained by requiring that the determinant of coefficients in the equations $\sum_j A_{ij}E_j=0$ $(i, j=x, y, z)$ be set equal to zero. The A_{ij} are given by

$$
\begin{aligned}
Axx &= (1 - iZ - \beta_z n)(n^2 - 1) + X(1 - \beta_z n)\\
Axy &= - iY_z(n^2 - 1)\\
Axz &= (1 - iZ - \beta_z n)\beta_x n - iY_y(1 - \beta_z n)\\
&\quad - iY_z\beta_y n\\
Ayx &= iY_z(n^2 - 1)\\
Ayy &= (1 - iZ - \beta_z n)(n^2 - 1) + X(1 - \beta_z n)\\
Ayz &= (1 - iZ - \beta_z n)\beta_y n + iY_x(1 - \beta_z n)\\
&\quad + iY_z\beta_x n\\
Azx &= X\beta_x n - iY_y(n^2 - 1)\\
Azy &= X\beta_y n + iY_x(n^2 - 1)\\
Azz &= -(1 - iZ - \beta_z n)(1 - \beta_z n) + X\\
&\quad + i(Y_x\beta_y - Y_y\beta_x)n.
\end{aligned}
$$

The correct dispersion relation is of eighth degree in n rather than sixth degree as obtained by Unz, or fourth degree as in the classical $(\beta=0)$ Appleton-Hartree equation. The author is presently carrying out analytical and numerical studies of several limiting cases. They will be reported on at a later date. However, a few conclusions which have been deduced so far are: 1) The attenuation factors of the new modes are proportional to β^{-1}; the new modes are therefore very heavily damped except in certain narrow ranges of X, Y and Z. 2) The drift velocity correction to the index of refraction for the classical modes is of relativistic order unless there is a component of velocity in the direction of propagation. 3) The mean motion of the plasma provides a mechanism for coupling of the longitudinal and transverse modes.

In the second paper,[2] the more restricted problem of plasma motion in the direction of propagation without an applied magnetic field was considered. Although proper account was taken of the convective term in the acceleration of the electron, the $v_0 \times B$ term was omitted. This was done because the author originally conceived of applying the results only at relatively low velocities. However upon reconsidering the problem, it was found that the omitted term is multiplied by $\nu_p^2(=\omega_p^2/\omega^2)$ and hence may be significant even at low velocities if the plasma frequency is much greater than the signal frequency. Hence, (7) of the second paper[2] should read

$$
Vr^3 + (\nu_c + i)r^2 + V(1 - \nu_p^2)r + \nu_c + i(1 - \nu_p^2) = 0.
$$

The original analysis is therefore only accurate at small values of ω_p/ω.

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Author's Reply[3]

This is to thank Dr. Epstein for his comments and for his interest in the problem.

In his original paper[4] the author used the method of solution suggested by Fano, Chu, and Adler,[5] namely, finding the constitutive relations for a stationary medium subject to the effective fields created by the motion of the plasma. By including the convective term of the electron acceleration, the constitutive relation for the drifting plasma may be found and used instead. Following Dr. Epstein's suggestion for small signal theory approximation, one could use in the constitutive relations given in Section III of the original paper:[4]

$$
\frac{d\overline{R}}{dt} = \left(\frac{\partial}{\partial t} + \bar{v}_0\cdot\nabla\right)\overline{R} \tag{1}
$$

where:

$$
\overline{R} = \overline{R}_0 e^{i(\omega t - kz)} \qquad \overline{R}_0 = \text{Const.} \tag{2a}
$$

$$
\bar{\beta} = \frac{\bar{v}_0}{c} = \beta_x \bar{i}_x + \beta_y \bar{i}_y + \beta_z \bar{i}_z. \tag{2b}
$$

Substituting (2) into (1) and taking $ck/\omega=n$, one obtains:

$$
\frac{d\overline{R}}{dt} = i\omega\gamma\overline{R} \tag{3a}
$$

$$
\gamma = 1 - n\beta_z = 1 - n\beta_L. \tag{3b}
$$

Taking (3) into account, one should use in the refractive index equation in Section IV and in (21) of the original paper,[4] the *effective values* of the plasma parameters, X_{eff}, $\overline{Y}_{\text{eff}}$, Z_{eff}, rather than the original values X, $\overline{Y}$, Z, where we define:

$$
X_{\text{eff}} = \frac{X}{\gamma^2} \tag{4a}
$$

$$
\overline{Y}_{\text{eff}} = \frac{\overline{Y}}{\gamma} \tag{4b}
$$

$$
Z_{\text{eff}} = \frac{Z}{\gamma} \tag{4c}
$$

In the particular case considered by Epstein,[6] one should take in (21) of the original paper[4] $\overline{Y}=0$, $\beta_T=0$ and one will obtain $CA^2=0$. Using effective values as suggested above, one obtains from $A=0$:

$$
(n^2 - 1)(1 - iZ_{\text{eff}}) + X_{\text{eff}}\gamma^2 = 0. \tag{5a}
$$

Assuming $n\neq 1$, one obtains from $C=0$:

$$
1 - X_{\text{eff}} - iZ_{\text{eff}} = 0. \tag{5b}
$$

Using (4), one obtains from (5) above:

$$
(n^2 - 1)(\gamma - iZ) + \gamma X = 0 \tag{6a}
$$

$$
\gamma^2 - iZ\gamma - X = 0 \tag{6b}
$$

where $\gamma = 1 - n\beta_L$. Eq. (6a) above is identical with the one given by Epstein in the first part of this discussion.

The author has written recently two additional papers[7,8] on the subject of propagation of electromagnetic waves in drifting media and they will be published in the near future.

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[3] Received October 29, 1962.

[4] H. Unz, "The magneto-ionic theory for drifting plasma," IRE TRANS. ON ANTENNAS AND PROPAGATION, vol. AP-10, pp. 459–464; July, 1962.

[5] R. M. Fano, L. J. Chu, and R. B. Adler, "Electromagnetic Fields, Energy and Forces," John Wiley and Sons, Inc., New York, N. Y., ch. 9; 1960.

[6] M. Epstein, "Effect of fluid motion on electromagnetic wave propagation," IRE TRANS. ON ANTENNAS AND PROPAGATION, vol. AP-10, pp. 645–646; September, 1962.

[7] H. Unz, "Drifting Plasma Magneto-Ionic Theory for Oblique Incidence," Antenna Lab., Ohio State University, Columbus, Ohio, Rept. No. 1116-24, pp. 1–22; October, 1962. Prepared under Contract No. AF 19(604)-7270, sponsored by the Detection Physics Lab., Electronics Res. Directorate, AFCR Labs., Office of Aerospace Res., U. S. AF, Bedford, Mass.

[8] H. Unz, "Electromagnetic Radiation in Drifting Tellegen Anisotropic Medium," Antenna Lab., Ohio State University, Columbus, Ohio, Rept. No. 1021-23, pp. 1–17; October, 1962. Prepared under Contract No. AF 33(616)-6782, sponsored by the Aeronautical Sys. Div., AF Sys Command, U. S. AF, Wright-Patterson AFB, Ohio.

Comments on "The Magneto-Ionic Theory for a Drifting Plasma"*

In a recent paper Unz[1] has attempted to develop a magneto-ionic theory for the case of a drifting plasma. His results, as stated, do not agree with those obtained by other workers.[2,3] The principal error in his exposition arises in his confusion of wave frequencies in the fixed and moving frames of reference.

In his development Unz quotes the following set of equations which describe the propagation of electromagnetic waves in a moving medium when viewed from a fixed frame of reference:

$$
\nabla \times \boldsymbol{H} - \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} = J_f + \frac{\partial \boldsymbol{P}}{\partial t} + \nabla \times (\boldsymbol{P} \times \boldsymbol{v}_0) \tag{1a}
$$

$$
\nabla \times \boldsymbol{E} + \mu_0 \frac{\partial \boldsymbol{H}}{\partial t} = -\frac{\partial}{\partial t}(\mu_0 \boldsymbol{M}) - \nabla \times (\mu_0 \boldsymbol{M} \times \boldsymbol{v}_0) \tag{1b}
$$

$$
\nabla \cdot (\epsilon_0 \boldsymbol{E}) = \rho_f - \nabla \cdot \boldsymbol{P} \tag{1c}
$$

$$
\nabla \cdot (\mu_0 \boldsymbol{H}) = -\nabla \cdot (\mu_0 \boldsymbol{M}) \tag{1d}
$$

in which ∇ is the divergence operator.

It is important to note that the above equations refer to fields E and H as seen in the fixed frame of reference, while the polarization vector $\boldsymbol{P}$ is evaluated in the frame in

* Received December 3, 1962.

[1] H. Unz, "The magneto-ionic theory for a drifting plasma," IRE TRANS. ON ANTENNAS AND PROPAGATION, vol. AP-10, pp. 459–464; July, 1962.

[2] V. A. Bailey, "Plane waves in an ionized gas with static electric and magnetic fields present," *Aust. J. Sci. Res.*, A, vol. 1, pp. 351–359; December, 1948.

[3] T. Bell and R. A. Helliwell, "Travelling wave amplification in the ionosphere," *Proc. Symp. on Physical Processes in the Sun-Earth Environment*, D.R.B., Canada, pp. 215–222; March, 1960.

which the electrons are at rest, *i.e.*, the moving frame.

In order to develop the constitutive relationship Unz quotes the following two additional formulas:

$$P = -NeR \tag{2b}$$

$$m\frac{d^2R}{dt^2} + m\nu\frac{dR}{dt} = -eE_T - e\mu_0(v_T \times H_T) \tag{9}$$

Once again it is important to note that the field quantities in (9) are those which should be measured in the fixed reference frame, while the displacement vector, R, by definition is measured in the rest frame of the electrons (*i.e.*, the moving frame of reference).

On the basis of small signal theory, Unz obtains from his (9) the following equation for the harmonic time-varying components:

$$-m\omega^2R + m\nu i\omega R = -eE - e\mu_0 i\omega(R \times H_0) - e\mu_0(v_0 \times H). \tag{11}$$

It is quite clear that the foregoing equation is correct only if we take the frequency, ω, to be that seen in the moving frame.

Unz apparently has overlooked this point, since elsewhere in his paper [(4a), (4b) and (4c) as derived from (1a), (1b) and (1c)] he utilizes the same quantity, ω, to denote frequency in the fixed frame, and nowhere does he attempt to differentiate between the two differing frequencies.

To see the importance of this error, we note that frequencies in the fixed frame are related to frequencies in the moving frame by the following well-known nonrelativistic transformation:

$$\omega_f(1 - n\beta) = \omega_m$$

in which ω_f and ω_m are the frequencies as seen in the fixed frame and moving frame, respectively; $\beta = v/c$ where v is the drift velocity of the electrons in the direction of the wave normal and n is the refractive index of the fixed frame evaluated at the frequency, ω_f. Consequently the quantities which Unz designates as X, Y, Z and n_s are in reality functions of the refractive index of the fixed frame, although Unz does not indicate this fact.

Thus Unz's theory, as stated, does not predict, as it should, the well-known phenomena of space charge waves or gyro resonance waves. On the other hand, these physically established phenomena exhibit themselves in the theory, if the correct frequency is utilized in (11) of Unz's paper.

Utilizing the correct frequency in (11), the following dispersion relation can be obtained for the case of longitudinal drift and longitudinal propagation, neglecting the independent space charge modes:

$$(n^2 - 1)(Y - 1 + \beta n) - X(1 - \beta n) = 0$$

in which all parameters are evaluated in the fixed frame. We note that this equation can easily be obtained by means of a Lorentz transformation applied to the refractive index in the moving frame. The above relation is in agreement with those obtained by other workers[2,3] and gives results which can differ significantly from the results of Unz's work.

As a result of Unz's error we must conclude that the validity of application of his theory is seriously in doubt.

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Author's Reply[4]

This is to thank Bell, Smith, and Brice for their discussion and for their interest in the problem. Their comment is essentially the same as the one communicated to the author by Dr. Epstein about three months ago and published recently.[5]

In his original paper[1] this author used the method of solution suggested by Fano, Chu and Adler,[6] namely, finding the constitutive relations for a stationary medium subject to the effective fields created by the motion of the plasma. If the constitutive relations of the drifting plasma are used instead, it can be shown[5] that the *effective values* of the plasma parameters X_{eff}, $\overline{Y}_{\text{eff}}$, Z_{eff} rather than the original values X, $\overline{Y}$, Z should be used in the refractive index equation in Section IV and in (21) of the original paper,[1] where we define

$$X_{\text{eff}} = \frac{X}{(1 - n\beta_L)^2} \tag{1a}$$

$$\overline{Y}_{\text{eff}} = \frac{\overline{Y}}{1 - n\beta_L} \tag{1b}$$

$$Z_{\text{eff}} = \frac{Z}{1 - n\beta_L}. \tag{1c}$$

For the particular case of longitudinal drift $\beta_T = 0$ and longitudinal propagation $Y_T = 0$, one obtains therefore from (21) of the original paper,[1] assuming $C_{\text{eff}} \neq 0$,

$$A_{\text{eff}} = (n^2 - 1)(1 - iZ_{\text{eff}}) + X_{\text{eff}}(1 - n\beta_L)^2 = \pm Y_{L\text{eff}}(n^2 - 1) \tag{2}$$

Substituting (1) into (2), one obtains

$$(n^2 - 1)(1 - n\beta_L \mp Y_L - iZ) + X(1 - n\beta_L) = 0 \tag{3}$$

where $+Y_L$ refers to the ordinary wave and $-Y_L$ refers to the extraordinary wave[7] in the case of no drift velocity $\beta_L = 0$. Eq. (3) is in agreement with the relationship given in the above discussion for the case of the extraordinary wave (negative sign) and no collisions $Z = 0$. Eq. (3) is also in agreement with a result given previously[5] for the case of no magnetic field $Y_L = 0$.

The author has attempted to explain some physical phenomena by using the constitutive relations for a stationary media. This was possible since it basically involved only quadratic equations for the refractive index n. It is believed that these physical phenomena could be explained along similar lines also when one uses the constitutive relations for the drifting media. However, as seen in (3), this will be a more difficult problem as it involves cubic equations for the refractive index n, which could be simplified under certain assumptions. The author is presently considering this problem and the results will be published elsewhere.

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[4] Received December 12, 1962.

[5] M. Epstein and H. Unz, "Comments on two papers dealing with electromagnetic wave propagation in moving plasma," IEEE Trans. on Antennas and Propagation, this issue, pp. 193–194.

[6] R. M. Fano, L. J. Chu, and R. B. Adler, "Electromagnetic Fields, Energy and Forces," John Wiley and Sons, Inc., New York, N. Y., ch. 9; 1960.

[7] J. A. Ratcliffe, "The Magneto-Ionic Theory and Its Application to the Ionosphere," Cambridge University Press, Cambridge, England, 1959.

Beam Position Errors vs Frequency of Antenna Arrays with Delay-Line Phasing*

The experimental Inertialess Steerable Communications Antenna (ISCAN) in La Plata, Md., employs long lengths of low-loss 70-ohm transmission line to feed the 24 array elements.[1] During the initial testing and alignment of the antenna the electrical length of the lines was measured. It became clear from the results that the phase velocity depends upon frequency. This means that the phase of the elements is in error for frequencies other than the frequency at which the adjustment is made, and raises the question of beam position errors which, in turn, may impose bandwidth limitations. These aspects have been investigated quantitatively by these writers. The main points are presented here because of the possible implications which they have on all antenna arrays using delay-line phasing.

The total inductance per unit length of a coaxial transmission line is composed of the internal inductance of the inner and outer conductors and the external inductance, associated with the flux external to the conductors.

The external inductance is

$$L_e = \frac{\mu}{2\pi}\cdot\ln\frac{b}{a}, \tag{1}$$

For the ISCAN lines $L_e = 0.285\ \mu H/\text{m}$.

The general expressions for the internal inductance are quite complicated, even in the simple case of a round wire and of a tubular outer conductor.[2] They simplify greatly,

* Received October 12, 1962.

[1] H. Brueckmann, J. Gruber, and C. Bramble, "ISCAN-Inertialess Steerable Communication Antenna," 1962 IRE International Convention Record, pt. 1, pp. 152–163.

[2] S. Ramo and J. R. Whinnery, "Fields and Waves in Modern Radio," John Wiley and Sons, Inc., New York, N. Y.; 1959. Plotted curves, given in this reference, facilitate the evaluation of the internal inductance for relatively low frequencies.